Inverse transforms; partial fractions reborn

Lesson 41 · Module 3 — Laplace Transforms · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Find the inverse Laplace transform of $Y(s) = \frac{2}{s^2-9}$.
2. Find the inverse Laplace transform of $Y(s) = \frac{s+5}{(s-1)^2}$.
3. Find the inverse Laplace transform of $Y(s) = \frac{1}{(s-2)(s^2+1)}$.

Solutions

1. Find the inverse Laplace transform of $Y(s) = \frac{2}{s^2-9}$.

   - Factor the denominator: $s^2 - 9 = (s-3)(s+3)$.
   - Set up partial fractions: $\frac{2}{(s-3)(s+3)} = \frac{A}{s-3} + \frac{B}{s+3}$.
   - Solve for constants: $2 = A(s+3) + B(s-3)$. For $s = 3$, $2 = 6A \implies A = 1/3$. For $s = -3$, $2 = -6B \implies B = -1/3$.
   - Rewrite $Y(s) = \frac{1/3}{s-3} - \frac{1/3}{s+3}$.
   - Apply inverse transforms: $y(t) = \frac{1}{3}e^{3t} - \frac{1}{3}e^{-3t}$.

   Answer: y(t) = \frac13(e^3t - e^-3t)

2. Find the inverse Laplace transform of $Y(s) = \frac{s+5}{(s-1)^2}$.

   - Set up partial fractions for repeated root: $\frac{s+5}{(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2}$.
   - Solve for constants: $s + 5 = A(s-1) + B$.
   - For $s = 1$, $6 = B$.
   - Comparing coefficients of s: $1 = A$.
   - Rewrite $Y(s) = \frac{1}{s-1} + \frac{6}{(s-1)^2}$.
   - Apply inverse transforms: $y(t) = e^t + 6te^t$.

   Answer: y(t) = (1 + 6t)e^t

3. Find the inverse Laplace transform of $Y(s) = \frac{1}{(s-2)(s^2+1)}$.

   - Set up partial fractions: $\frac{1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$.
   - Solve for constants: $1 = A(s^2+1) + (Bs+C)(s-2)$.
   - For $s = 2$, $1 = 5A \implies A = 1/5$.
   - For $s = 0$, $1 = A - 2C \implies 1 = 1/5 - 2C \implies 2C = -4/5 \implies C = -2/5$.
   - Comparing s^2 coefficients: $0 = A + B \implies B = -1/5$.
   - Rewrite $Y(s) = \frac{1/5}{s-2} + \frac{-1/5s-2/5}{s^2+1}$.
   - Split the quadratic term: $Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1}$.
   - Apply inverse transforms: $y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t)$.

   Answer: y(t) = \frac15(e^2t - \cos(t) - 2\sin(t))
