The Laplace Transform: Turning Calculus into Algebra

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Practice problems

1. Find the Laplace transform of $f(t) = 3e^{-4t} + 2t$.
2. Given that $\mathcal{L}\{f(t)\} = F(s)$, find the expression for $\mathcal{L}\{f'(t)\}$ if $f(0) = 5$.
3. Find the Laplace transform of $f(t) = te^{2t}$. (Hint: Use the integral definition $\int_0^\infty e^{-st} f(t)dt$ and integration by parts).

Solutions

1. Find the Laplace transform of $f(t) = 3e^{-4t} + 2t$.

   - Use linearity: $\mathcal{L}\{3e^{-4t} + 2t\} = 3\mathcal{L}\{e^{-4t}\} + 2\mathcal{L}\{t\}$.
   - Apply the exponential rule with $a = -4$: $\mathcal{L}\{e^{-4t}\} = \frac{1}{s-(-4)} = \frac{1}{s+4}$.
   - Apply the power rule: $\mathcal{L}\{t\} = \frac{1}{s^2}$.
   - Combine the results: $3(\frac{1}{s+4}) + 2(\frac{1}{s^2})$.

   Answer: $\frac{3}{s+4} + \frac{2}{s^2}$

2. Given that $\mathcal{L}\{f(t)\} = F(s)$, find the expression for $\mathcal{L}\{f'(t)\}$ if $f(0) = 5$.

   - Recall the derivative formula: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$.
   - Substitute the given initial value $f(0) = 5$ into the formula.
   - The expression becomes $sF(s) - 5$.

   Answer: $sF(s) - 5$

3. Find the Laplace transform of $f(t) = te^{2t}$. (Hint: Use the integral definition $\int_0^\infty e^{-st} f(t)dt$ and integration by parts).

   - Set up the integral: $\int_0^\infty e^{-st}(te^{2t})dt = \int_0^\infty te^{-(s-2)t}dt$.
   - Use integration by parts with $u = t$ and $dv = e^{-(s-2)t}dt$.
   - Then $du = dt$ and $v = \frac{-1}{s-2}e^{-(s-2)t}$.
   - The boundary term $[uv]_0^\infty$ is 0 for $s > 2$.
   - The remaining integral is $\int_0^\infty \frac{1}{s-2}e^{-(s-2)t}dt$.
   - Integrating again gives $\frac{1}{s-2}[\frac{-1}{s-2}e^{-(s-2)t}]_0^\infty = \frac{1}{(s-2)^2}$.

   Answer: $\frac{1}{(s-2)^2}$
