Module 2 Review and Self-Test

Practice problems

1. Find the general solution to the differential equation $y'' - 3y' + 2y = 4t^2$.
2. Solve the initial value problem $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 2$.
3. Use variation of parameters to find a particular solution for $y'' + y = \sec(t)$.

Solutions

1. Find the general solution to the differential equation $y'' - 3y' + 2y = 4t^2$.

   - Solve the homogeneous equation $y''-3y'+2y = 0$. The characteristic equation is $r^2-3r+2 = 0$, which factors to $(r-1)(r-2) = 0$. Thus $y_h = c_1e^t + c_2e^{2t}$.
   - Guess a particular solution of the form $y_p = At^2 + Bt + C$.
   - Compute derivatives: $y_p' = 2At + B$ and $y_p'' = 2A$.
   - Substitute into the ODE: $2A - 3(2At + B) + 2(At^2 + Bt + C) = 4t^2$.
   - Group terms: $2At^2 + (2B - 6A)t + (2A - 3B + 2C) = 4t^2$.
   - Solve the system: $2A = 4 \implies A = 2$; $2B - 6(2) = 0 \implies B = 6$; $2(2) - 3(6) + 2C = 0 \implies 2C = 14 \implies C = 7$.
   - Combine: $y = c_1e^t + c_2e^{2t} + 2t^2 + 6t + 7$.

   Answer: $y = c_1e^t + c_2e^{2t} + 2t^2 + 6t + 7$

2. Solve the initial value problem $y'' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 2$.

   - The characteristic equation is $r^2 + 4 = 0$, so $r = \pm 2i$.
   - The general solution is $y = c_1 \cos(2t) + c_2 \sin(2t)$.
   - Apply $y(0) = 3$: $c_1 \cos(0) + c_2 \sin(0) = 3 \implies c_1 = 3$.
   - Compute $y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$.
   - Apply $y'(0) = 2$: $-2(3)\sin(0) + 2c_2 \cos(0) = 2 \implies 2c_2 = 2 \implies c_2 = 1$.
   - The solution is $y = 3\cos(2t) + \sin(2t)$.

   Answer: $y = 3\cos(2t) + \sin(2t)$

3. Use variation of parameters to find a particular solution for $y'' + y = \sec(t)$.

   - Homogeneous solutions are $y_1 = \cos(t)$ and $y_2 = \sin(t)$.
   - The Wronskian is $W = \cos(t)\cos(t) - \sin(t)(-\sin(t)) = 1$.
   - The particular solution is $y_p = -\cos(t)\int \sin(t)\sec(t)dt + \sin(t)\int \cos(t)\sec(t)dt$.
   - Simplify the integrals: $\int \sin(t)\sec(t)dt = \int \tan(t)dt = -\ln|\cos(t)|$.
   - And $\int \cos(t)\sec(t)dt = \int 1dt = t$.
   - Combine: $y_p = -\cos(t)(-\ln|\cos(t)|) + \sin(t)(t) = \cos(t)\ln|\cos(t)| + t\sin(t)$.

   Answer: $y_p = \cos(t)\ln|\cos(t)| + t\sin(t)$
