Higher-order linear equations

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Practice problems

1. Find the general solution to the third-order linear ODE $y''' - 5y'' + 8y' - 4y = 0$.
2. Solve the ODE $y''' + 3y'' + 3y' + y = 0$ with initial conditions $y(0) = 1, y'(0) = 0, y''(0) = 0$.
3. Find the general solution to the fourth-order ODE $y'''' + 2y'' + y = 0$.

Solutions

1. Find the general solution to the third-order linear ODE $y''' - 5y'' + 8y' - 4y = 0$.

   - Write the characteristic equation: $r^3 - 5r^2 + 8r - 4 = 0$.
   - Test for roots: $r = 1$ gives $1 - 5 + 8 - 4 = 0$, so $(r-1)$ is a factor.
   - Divide to get $(r-1)(r^2 - 4r + 4) = 0$, which is $(r-1)(r-2)^2 = 0$.
   - The roots are $r = 1$ (distinct) and $r = 2$ (multiplicity 2).
   - Construct the general solution using the repeated root rule.

   Answer: $y(t) = C_1 e^t + (C_2 + C_3 t)e^{2t}$

2. Solve the ODE $y''' + 3y'' + 3y' + y = 0$ with initial conditions $y(0) = 1, y'(0) = 0, y''(0) = 0$.

   - The characteristic equation is $r^3 + 3r^2 + 3r + 1 = 0$, which is $(r+1)^3 = 0$.
   - The root $r = -1$ has multiplicity 3, so $y(t) = (C_1 + C_2 t + C_3 t^2)e^{-t}$.
   - Use $y(0) = 1$ implies $C_1 = 1$.
   - Differentiate: $y'(t) = (C_2 + 2C_3 t)e^{-t} - (C_1 + C_2 t + C_3 t^2)e^{-t}$.
   - Use $y'(0) = 0$ implies $C_2 - C_1 = 0$ so $C_2 = 1$.
   - Differentiate again and use $y''(0) = 0$ implies $2C_3 - 2C_2 + C_1 = 0$ so $2C_3 - 2 + 1 = 0$ so $C_3 = 1/2$.

   Answer: $y(t) = (1 + t + \frac{1}{2}t^2)e^{-t}$

3. Find the general solution to the fourth-order ODE $y'''' + 2y'' + y = 0$.

   - The characteristic equation is $r^4 + 2r^2 + 1 = 0$.
   - This is a perfect square: $(r^2 + 1)^2 = 0$.
   - The roots are $r = i$ and $r = -i$, each with multiplicity 2.
   - For the first pair of complex roots, we have $\cos(t)$ and $\sin(t)$.
   - Because they are repeated, we multiply by t for the second pair: $t\cos(t)$ and $t\sin(t)$.

   Answer: $y(t) = C_1 \cos(t) + C_2 \sin(t) + C_3 t \cos(t) + C_4 t \sin(t)$
