Forced oscillations and resonance I — the phenomenon

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Practice problems

1. Find the particular solution $y_p(t)$ for the differential equation $y'' + y = \cos(t)$.
2. Determine if the equation $y'' + 16y = 5\sin(3t)$ exhibits resonance. If not, find the particular solution $y_p(t)$.
3. A system is described by $y'' + 25y = 20\cos(5t)$. Find the general solution $y(t)$ given initial conditions $y(0) = 0$ and $y'(0) = 0$.

Solutions

1. Find the particular solution $y_p(t)$ for the differential equation $y'' + y = \cos(t)$.

   - The homogeneous equation $y'' + y = 0$ has natural frequency $\omega_0 = 1$.
   - The forcing function $\cos(t)$ has frequency $\omega = 1$. Since $\omega = \omega_0$, resonance occurs.
   - Guess $y_p(t) = t(A\cos(t) + B\sin(t))$.
   - Compute $y_p' = A\cos(t) + B\sin(t) + t(-A\sin(t) + B\cos(t))$.
   - Compute $y_p'' = -2A\sin(t) + 2B\cos(t) - t(A\cos(t) + B\sin(t))$.
   - Substitute into $y'' + y = \cos(t)$: $-2A\sin(t) + 2B\cos(t) = \cos(t)$.
   - Match coefficients: $-2A = 0 \implies A = 0$ and $2B = 1 \implies B = 1/2$.

   Answer: y_p(t) = \frac12t \sin(t)

2. Determine if the equation $y'' + 16y = 5\sin(3t)$ exhibits resonance. If not, find the particular solution $y_p(t)$.

   - Natural frequency $\omega_0 = \sqrt{16} = 4$.
   - Forcing frequency $\omega = 3$. Since $3 \neq 4$, there is no resonance.
   - Guess $y_p(t) = A\cos(3t) + B\sin(3t)$.
   - Substitute into ODE: $(-9A + 16A)\cos(3t) + (-9B + 16B)\sin(3t) = 5\sin(3t)$.
   - Simplify: $7A\cos(3t) + 7B\sin(3t) = 5\sin(3t)$.
   - Solve: $7A = 0 \implies A = 0$ and $7B = 5 \implies B = 5/7$.

   Answer: y_p(t) = \frac57 \sin(3t)

3. A system is described by $y'' + 25y = 20\cos(5t)$. Find the general solution $y(t)$ given initial conditions $y(0) = 0$ and $y'(0) = 0$.

   - Homogeneous solution: $y_h(t) = C_1\cos(5t) + C_2\sin(5t)$.
   - Resonance occurs since $\omega = \omega_0 = 5$. Guess $y_p(t) = t(A\cos(5t) + B\sin(5t))$.
   - Substitute into ODE: $y_p'' + 25y_p = -10A\sin(5t) + 10B\cos(5t) = 20\cos(5t)$.
   - Solve: $-10A = 0 \implies A = 0$ and $10B = 20 \implies B = 2$. So $y_p(t) = 2t\sin(5t)$.
   - General solution: $y(t) = C_1\cos(5t) + C_2\sin(5t) + 2t\sin(5t)$.
   - Apply $y(0) = 0$: $C_1(1) + C_2(0) + 0 = 0 \implies C_1 = 0$.
   - Apply $y'(0) = 0$: $y'(t) = 5C_2\cos(5t) + 2\sin(5t) + 10t\cos(5t)$. $5C_2(1) + 0 + 0 = 0 \implies C_2 = 0$.

   Answer: y(t) = 2t \sin(5t)
