Variation of Parameters

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Practice problems

1. Find a particular solution $y_p(t)$ for the differential equation $y'' + 4y = \sec(2t)$.
2. Solve the nonhomogeneous equation $y'' - y = \frac{e^t}{t}$ for $t > 0$.
3. Find the general solution to $y'' + y = \tan(t)$ given the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solutions

1. Find a particular solution $y_p(t)$ for the differential equation $y'' + 4y = \sec(2t)$.

   - The homogeneous solutions are $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$.
   - The Wronskian is $W = \cos(2t)(2\cos(2t)) - \sin(2t)(-2\sin(2t)) = 2(\cos^2(2t) + \sin^2(2t)) = 2$.
   - Compute $u_1' = \frac{-\sin(2t)\sec(2t)}{2} = -\frac{1}{2}\tan(2t)$.
   - Integrate $u_1 = \int -\frac{1}{2}\tan(2t)dt = \frac{1}{4}\ln|\cos(2t)|$.
   - Compute $u_2' = \frac{\cos(2t)\sec(2t)}{2} = \frac{1}{2}$.
   - Integrate $u_2 = \int \frac{1}{2}dt = \frac{1}{2}t$.
   - Assemble $y_p = \frac{1}{4}\ln|\cos(2t)|\cos(2t) + \frac{1}{2}t\sin(2t)$.

   Answer: y_p(t) = \frac14\cos(2t)\ln—\cos(2t)— + \frac12t\sin(2t)

2. Solve the nonhomogeneous equation $y'' - y = \frac{e^t}{t}$ for $t > 0$.

   - Homogeneous solutions are $y_1 = e^t$ and $y_2 = e^{-t}$.
   - Wronskian $W = e^t(-e^{-t}) - e^{-t}(e^t) = -1 - 1 = -2$.
   - Compute $u_1' = \frac{-e^{-t}(e^t/t)}{-2} = \frac{1}{2t}$.
   - Integrate $u_1 = \frac{1}{2}\ln(t)$.
   - Compute $u_2' = \frac{e^t(e^t/t)}{-2} = -\frac{e^{2t}}{2t}$.
   - The integral for u_2 cannot be expressed in elementary functions, so we leave it as $u_2 = -\int \frac{e^{2t}}{2t}dt$.
   - Assemble $y_p = \frac{1}{2}e^t\ln(t) - e^{-t}\int \frac{e^{2t}}{2t}dt$.

   Answer: y_p(t) = \frac12e^t\ln(t) - e^-t\int \frace^2t2t dt

3. Find the general solution to $y'' + y = \tan(t)$ given the initial conditions $y(0) = 0$ and $y'(0) = 1$.

   - From the lesson example, the general solution is $y(t) = c_1\cos(t) + c_2\sin(t) - \cos(t)\ln|\sec(t) + \tan(t)|$.
   - Apply $y(0) = 0$: $0 = c_1(1) + c_2(0) - (1)\ln|1 + 0| \implies c_1 = 0$.
   - Find $y'(t) = c_2\cos(t) + \sin(t)\ln|\sec(t) + \tan(t)| - \cos(t)\frac{\sec(t)\tan(t)+\sec^2(t)}{\sec(t)+\tan(t)}$.
   - Simplify the fraction: $\frac{\sec(t)(\tan(t)+\sec(t))}{\sec(t)+\tan(t)} = \sec(t)$.
   - So $y'(t) = c_2\cos(t) + \sin(t)\ln|\sec(t) + \tan(t)| - \cos(t)\sec(t) = c_2\cos(t) + \sin(t)\ln|\sec(t) + \tan(t)| - 1$.
   - Apply $y'(0) = 1$: $1 = c_2(1) + 0 - 1 \implies c_2 = 2$.
   - The final solution is $y(t) = 2\sin(t) - \cos(t)\ln|\sec(t) + \tan(t)|$.

   Answer: y(t) = 2\sin(t) - \cos(t)\ln—\sec(t) + \tan(t)—