Undetermined coefficients II — trickier right-hand sides

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Practice problems

1. Find the general solution to $y'' + 9y = \cos(3t)$.
2. Find the particular solution y_p for $y'' - 4y' + 4y = te^{2t}$.
3. Solve $y'' + y = 2\cos(t) + e^t$.

Solutions

1. Find the general solution to $y'' + 9y = \cos(3t)$.

   - The homogeneous equation $y''+9y = 0$ has characteristic roots $r = \pm 3i$, so $y_h = c_1 \cos(3t) + c_2 \sin(3t)$.
   - The forcing function $\cos(3t)$ is a solution to the homogeneous equation, indicating resonance.
   - The guess for the particular solution is $y_p = t(A\cos(3t) + B\sin(3t))$.
   - Computing derivatives: $y_p' = (A\cos(3t) + B\sin(3t)) + t(-3A\sin(3t) + 3B\cos(3t))$.
   - Computing y_p'' and substituting into $y'' + 9y = \cos(3t)$ leads to $-6A\sin(3t) + 6B\cos(3t) = \cos(3t)$.
   - Matching coefficients: $-6A = 0 \implies A = 0$ and $6B = 1 \implies B = 1/6$.

   Answer: y(t) = c_1 \cos(3t) + c_2 \sin(3t) + \frac16t \sin(3t)

2. Find the particular solution y_p for $y'' - 4y' + 4y = te^{2t}$.

   - The homogeneous equation $y''-4y'+4y = 0$ has a repeated root $r = 2$, so $y_h = c_1e^{2t} + c_2te^{2t}$.
   - The forcing function te^{2t} is already part of the homogeneous solution.
   - The standard guess for te^{2t} would be $(At + B)e^{2t}$.
   - Because both e^{2t} and te^{2t} are in y_h, we must multiply the guess by t^2.
   - The correct guess is $y_p = t^2(At + B)e^{2t} = (At^3 + Bt^2)e^{2t}$.
   - Substituting into the ODE and solving for coefficients yields $A = 1/6$ and $B = 0$.

   Answer: y_p(t) = \frac16t^3 e^2t

3. Solve $y'' + y = 2\cos(t) + e^t$.

   - Homogeneous solution: $y_h = c_1 \cos(t) + c_2 \sin(t)$.
   - For $f_1(t) = 2\cos(t)$, resonance occurs. Guess $y_{p1} = t(A\cos(t) + B\sin(t))$. Substituting gives $y_{p1} = t\sin(t)$.
   - For $f_2(t) = e^t$, no resonance. Guess $y_{p2} = Ce^t$. Substituting gives $C + C = 1 \implies C = 1/2$.
   - The total particular solution is $y_p = t\sin(t) + \frac{1}{2}e^t$.

   Answer: y(t) = c_1 \cos(t) + c_2 \sin(t) + t \sin(t) + \frac12e^t

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