Nonhomogeneous equations: undetermined coefficients I

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Practice problems

1. Find the general solution to the differential equation $y'' - 4y' + 3y = 6t$.
2. Find the general solution to $y'' - 5y' + 6y = 2e^{4t}$.
3. Find the general solution to $y'' - 3y' + 2y = e^{2t}$.

Solutions

1. Find the general solution to the differential equation $y'' - 4y' + 3y = 6t$.

    - Solve the homogeneous equation $y''-4y'+3y = 0$. The characteristic equation is $r^2-4r+3 = 0$, which factors to $(r-1)(r-3) = 0$, giving $y_h = c_1e^t + c_2e^{3t}$.
    - Guess a particular solution of the form $y_p = At + B$.
    - Compute derivatives: $y_p' = A$ and $y_p'' = 0$.
    - Substitute into the ODE: $0 - 4(A) + 3(At + B) = 6t$.
    - Group terms: $3At + (3B - 4A) = 6t$.
    - Match coefficients: $3A = 6 \to A = 2$. Then $3B - 4(2) = 0 \to 3B = 8 \to B = 8/3$.
    - Combine: $y(t) = c_1e^t + c_2e^{3t} + 2t + 8/3$.

    Answer: $y(t) = c_1e^t + c_2e^{3t} + 2t + \frac{8}{3}$

2. Find the general solution to $y'' - 5y' + 6y = 2e^{4t}$.

    - Solve the homogeneous equation $y''-5y'+6y = 0$. The characteristic equation is $r^2-5r+6 = 0$, which factors to $(r-2)(r-3) = 0$, giving $y_h = c_1e^{2t} + c_2e^{3t}$.
    - The forcing function is $2e^{4t}$. Since e^{4t} is not in y_h, guess $y_p = Ae^{4t}$.
    - Compute derivatives: $y_p' = 4Ae^{4t}$ and $y_p'' = 16Ae^{4t}$.
    - Substitute into the ODE: $16Ae^{4t} - 5(4Ae^{4t}) + 6(Ae^{4t}) = 2e^{4t}$.
    - Simplify: $(16 - 20 + 6)Ae^{4t} = 2e^{4t} \to 2Ae^{4t} = 2e^{4t}$.
    - Solve for A: $2A = 2 \to A = 1$.
    - Combine: $y(t) = c_1e^{2t} + c_2e^{3t} + e^{4t}$.

    Answer: $y(t) = c_1e^{2t} + c_2e^{3t} + e^{4t}$

3. Find the general solution to $y'' - 3y' + 2y = e^{2t}$.

    - Solve the homogeneous equation $y''-3y'+2y = 0$. The characteristic equation is $r^2-3r+2 = 0$, which factors to $(r-1)(r-2) = 0$, giving $y_h = c_1e^t + c_2e^{2t}$.
    - The forcing function is e^{2t}. Since e^{2t} is already in y_h, the guess Ae^{2t} will fail. Multiply by t and guess $y_p = Ate^{2t}$.
    - Compute derivatives: $y_p' = Ae^{2t}+2Ate^{2t}$ and $y_p'' = 2Ae^{2t}+2Ae^{2t}+4Ate^{2t} = 4Ae^{2t}+4Ate^{2t}$.
    - Substitute into the ODE: $(4Ae^{2t} + 4Ate^{2t}) - 3(Ae^{2t} + 2Ate^{2t}) + 2(Ate^{2t}) = e^{2t}$.
    - Simplify: $(4A - 3A)e^{2t} + (4At - 6At + 2At)e^{2t} = e^{2t} \to Ae^{2t} = e^{2t}$.
    - Solve for A: $A = 1$.
    - Combine: $y(t) = c_1e^t + c_2e^{2t} + te^{2t}$.

    Answer: $y(t) = c_1e^t + c_2e^{2t} + te^{2t}$