The harmonic oscillator II — damping regimes side by side

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Practice problems

1. A system is described by the ODE $2y'' + 8y' + 8y = 0$. Determine the damping regime and find the general solution.
2. For a mass $m = 1$ and spring constant $k = 9$, what value of the damping coefficient b results in critical damping?
3. Solve the IVP $y'' + 2y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 0$. Identify the regime.

Solutions

1. A system is described by the ODE $2y'' + 8y' + 8y = 0$. Determine the damping regime and find the general solution.

   - Divide by 2 to get $y'' + 4y' + 4y = 0$.
   - The characteristic equation is $r^2 + 4r + 4 = 0$.
   - Factor as $(r + 2)^2 = 0$, giving a repeated root $r = -2$.
   - Since the discriminant $4^2 - 4(1)(4) = 0$, the system is critically damped.
   - The general solution is $y(t) = (c_1 + c_2 t)e^{-2t}$.

   Answer: Critically damped; $y(t) = (c_1 + c_2 t)e^{-2t}$

2. For a mass $m = 1$ and spring constant $k = 9$, what value of the damping coefficient b results in critical damping?

   - Critical damping occurs when the discriminant $b^2 - 4mk = 0$.
   - Substitute $m = 1$ and $k = 9$: $b^2 - 4(1)(9) = 0$.
   - Solve for b: $b^2 = 36$.
   - Since b must be positive, $b = 6$.

   Answer: $b = 6$

3. Solve the IVP $y'' + 2y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 0$. Identify the regime.

   - Characteristic equation: $r^2 + 2r + 5 = 0$.
   - Roots: $r = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$.
   - Discriminant is $-16 < 0$, so it is under-damped.
   - General solution: $y(t) = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$.
   - Apply $y(0) = 1$: $1 = e^0(c_1(1) + c_2(0)) \implies c_1 = 1$.
   - Derivative: $y'(t) = -e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t))$.
   - Apply $y'(0) = 0$: $0 = -1(c_1) + 1(2c_2) \implies 0 = -1 + 2c_2 \implies c_2 = 1/2$.
   - Final solution: $y(t) = e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t))$.

   Answer: Under-damped; $y(t) = e^{-t}(\cos(2t) + \frac{1}{2}\sin(2t))$
