Complex roots — oscillation emerges

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Practice problems

1. Find the general solution to the differential equation $y'' + 6y' + 25y = 0$.
2. Solve the initial value problem $y'' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 3$.
3. A system is described by $y'' + 2y' + 2y = 0$. If the solution is $y(t) = e^{-t}\sin(t)$, verify that it satisfies the ODE.

Solutions

1. Find the general solution to the differential equation $y'' + 6y' + 25y = 0$.
   - Write the characteristic equation: $r^2 + 6r + 25 = 0$.
   - Use the quadratic formula: $r = \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm \sqrt{-64}}{2}$.
   - Simplify the roots: $r = \frac{-6 \pm 8i}{2} = -3 \pm 4i$.
   - Identify $\alpha = -3$ and $\beta = 4$.
   - Substitute into the general form: $y(t) = e^{-3t}(c_1 \cos(4t) + c_2 \sin(4t))$.

   Answer: $y(t) = e^{-3t}(c_1 \cos(4t) + c_2 \sin(4t))$

2. Solve the initial value problem $y'' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 3$.
   - Characteristic equation: $r^2 + 9 = 0$, so $r = \pm 3i$.
   - Here $\alpha = 0$ and $\beta = 3$, so $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$.
   - Apply $y(0) = 2$: $c_1 \cos(0) + c_2 \sin(0) = 2 \implies c_1 = 2$.
   - Find the derivative: $y'(t) = -3c_1 \sin(3t) + 3c_2 \cos(3t)$.
   - Apply $y'(0) = 3$: $-3(2)(0) + 3c_2(1) = 3 \implies 3c_2 = 3 \implies c_2 = 1$.

   Answer: $y(t) = 2\cos(3t) + \sin(3t)$

3. A system is described by $y'' + 2y' + 2y = 0$. If the solution is $y(t) = e^{-t}\sin(t)$, verify that it satisfies the ODE.
   - Compute $y'(t)$: Using product rule, $y' = -e^{-t}\sin(t) + e^{-t}\cos(t)$.
   - Compute $y''(t)$: $y'' = (e^{-t}\sin(t) - e^{-t}\cos(t)) + (-e^{-t}\cos(t) - e^{-t}\sin(t)) = -2e^{-t}\cos(t)$.
   - Substitute into $y'' + 2y' + 2y$: $(-2e^{-t}\cos(t)) + 2(-e^{-t}\sin(t) + e^{-t}\cos(t)) + 2(e^{-t}\sin(t))$.
   - Simplify: $-2e^{-t}\cos(t) - 2e^{-t}\sin(t) + 2e^{-t}\cos(t) + 2e^{-t}\sin(t) = 0$.

   Answer: The expression simplifies to 0, so the solution is verified.
