Repeated roots — critical damping

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Practice problems

1. Find the general solution for the differential equation $y'' + 10y' + 25y = 0$.
2. Solve the initial value problem $y'' + 4y' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 1$.
3. A system is described by $y'' + 2\beta y' + \beta^2 y = 0$. If the system is released from $y(0) = y_0$ with zero initial velocity $y'(0) = 0$, show that the solution is $y(t) = y_0(1 + \beta t)e^{-\beta t}$.

Solutions

1. Find the general solution for the differential equation $y'' + 10y' + 25y = 0$.

   - Write the characteristic equation: $r^2 + 10r + 25 = 0$.
   - Factor the quadratic: $(r + 5)^2 = 0$.
   - Identify the repeated root: $r = -5$.
   - Apply the general form for repeated roots: $y(t) = (c_1 + c_2 t)e^{-5t}$.

   Answer: $y(t) = (c_1 + c_2 t)e^{-5t}$

2. Solve the initial value problem $y'' + 4y' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 1$.

   - Characteristic equation: $r^2 + 4r + 4 = (r + 2)^2 = 0$, so $r = -2$.
   - General solution: $y(t) = (c_1 + c_2 t)e^{-2t}$.
   - Use $y(0) = 3$: $c_1 e^0 = 3 \implies c_1 = 3$.
   - Find derivative: $y'(t) = c_2 e^{-2t} - 2(c_1 + c_2 t)e^{-2t}$.
   - Use $y'(0) = 1$: $c_2 - 2c_1 = 1 \implies c_2 - 6 = 1 \implies c_2 = 7$.
   - Final solution: $y(t) = (3 + 7t)e^{-2t}$.

   Answer: $y(t) = (3 + 7t)e^{-2t}$

3. A system is described by $y'' + 2\beta y' + \beta^2 y = 0$. If the system is released from $y(0) = y_0$ with zero initial velocity $y'(0) = 0$, show that the solution is $y(t) = y_0(1 + \beta t)e^{-\beta t}$.

   - Characteristic equation: $r^2 + 2\beta r + \beta^2 = (r + \beta)^2 = 0$, so $r = -\beta$.
   - General solution: $y(t) = (c_1 + c_2 t)e^{-\beta t}$.
   - Apply $y(0) = y_0$: $c_1 e^0 = y_0 \implies c_1 = y_0$.
   - Find derivative: $y'(t) = c_2 e^{-\beta t} - \beta(c_1 + c_2 t)e^{-\beta t}$.
   - Apply $y'(0) = 0$: $c_2 - \beta c_1 = 0 \implies c_2 = \beta y_0$.
   - Substitute constants: $y(t) = (y_0 + \beta y_0 t)e^{-\beta t} = y_0(1 + \beta t)e^{-\beta t}$.

   Answer: $y(t) = y_0(1 + \beta t)e^{-\beta t}$
