Homogeneous constant-coefficient equations: the characteristic equation

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Practice problems

1. Find the general solution to the differential equation $y'' + 7y' + 10y = 0$.
2. Solve the initial value problem $y'' - 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = 5$.
3. Determine the values of k for which the general solution of $y'' + ky' + 4y = 0$ consists of two distinct real exponential terms.

Solutions

1. Find the general solution to the differential equation $y'' + 7y' + 10y = 0$.

   - Write the characteristic equation: $r^2 + 7r + 10 = 0$.
   - Factor the quadratic: $(r + 2)(r + 5) = 0$.
   - Identify the roots: $r_1 = -2$ and $r_2 = -5$.
   - Construct the general solution using the linear combination of exponentials.

   Answer: $y(t) = c_1 e^{-2t} + c_2 e^{-5t}$

2. Solve the initial value problem $y'' - 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = 5$.

   - Characteristic equation: $r^2 - 4r + 3 = 0$, which factors to $(r - 1)(r - 3) = 0$.
   - General solution: $y(t) = c_1 e^t + c_2 e^{3t}$.
   - Use $y(0) = 2$: $c_1 + c_2 = 2$.
   - Find $y'(t) = c_1 e^t + 3c_2 e^{3t}$ and use $y'(0) = 5$: $c_1 + 3c_2 = 5$.
   - Subtract the first equation from the second: $2c_2 = 3$, so $c_2 = 1.5$.
   - Substitute back: $c_1 + 1.5 = 2$, so $c_1 = 0.5$.

   Answer: $y(t) = 0.5e^t + 1.5e^{3t}$

3. Determine the values of k for which the general solution of $y'' + ky' + 4y = 0$ consists of two distinct real exponential terms.

   - The characteristic equation is $r^2 + kr + 4 = 0$.
   - For the roots to be distinct and real, the discriminant must be strictly positive: $b^2 - 4ac > 0$.
   - Substitute the coefficients: $k^2 - 4(1)(4) > 0$.
   - Solve the inequality: $k^2 - 16 > 0$.
   - This means $k^2 > 16$, which implies $k > 4$ or $k < -4$.

   Answer: k must be in the interval $(-\infty, -4) \cup (4, \infty)$
