Linear independence of solutions; the Wronskian

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Practice problems

1. Determine if $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ are linearly independent by computing their Wronskian.
2. Given the ODE $y'' - 4y = 0$, verify that $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ form a fundamental set of solutions.
3. Find the general solution to $y'' + 9y = 0$ given that $y_1 = \sin(3t)$ and $y_2 = \cos(3t)$ are solutions.

Solutions

1. Determine if $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ are linearly independent by computing their Wronskian.
   - Find the derivatives: $y_1' = 2e^{2t}$ and $y_2' = -2e^{-2t}$.
   - Set up the Wronskian: $W = (e^{2t})(-2e^{-2t}) - (e^{-2t})(2e^{2t})$.
   - Simplify the terms: $W = -2(e^0) - 2(e^0) = -2 - 2 = -4$.
   - Since $W = -4$, which is never zero, the functions are linearly independent.

   Answer: Linearly independent, W = -4

2. Given the ODE $y'' - 4y = 0$, verify that $y_1 = e^{2t}$ and $y_2 = e^{-2t}$ form a fundamental set of solutions.
   - Verify y_1: $y_1'' = 4e^{2t}$, so $4e^{2t} - 4(e^{2t}) = 0$. Correct.
   - Verify y_2: $y_2'' = 4e^{-2t}$, so $4e^{-2t} - 4(e^{-2t}) = 0$. Correct.
   - Compute Wronskian: $W = (e^{2t})(-2e^{-2t}) - (e^{-2t})(2e^{2t}) = -4$.
   - Since both are solutions and $W \neq 0$, they form a fundamental set.

   Answer: They form a fundamental set because both solve the ODE and $W = -4 \neq 0$.

3. Find the general solution to $y'' + 9y = 0$ given that $y_1 = \sin(3t)$ and $y_2 = \cos(3t)$ are solutions.
   - Compute the Wronskian to ensure independence: $W = (\sin 3t)(-3\sin 3t) - (\cos 3t)(3\cos 3t)$.
   - Simplify: $W = -3(\sin^2 3t + \cos^2 3t) = -3(1) = -3$.
   - Since $W \neq 0$, the solutions are linearly independent.
   - The general solution is the linear combination $y(t) = c_1 \sin(3t) + c_2 \cos(3t)$.

   Answer: $y(t) = c_1 \sin(3t) + c_2 \cos(3t)$
