Module 1 review + self-test

Practice problems

1. Solve the initial value problem $y' = y\tan(t)$ with $y(0) = 5$.
2. A tank contains 50 liters of water with 100 grams of salt. Brine with 2 grams per liter flows in at 3 liters per minute, and the mixture flows out at 3 liters per minute. Find the ODE for the salt $y(t)$ and the equilibrium value.
3. Consider the autonomous equation $y' = y(y-2)(y-4)$. Identify all equilibria and classify their stability.

Solutions

1. Solve the initial value problem $y' = y\tan(t)$ with $y(0) = 5$.

   - Separate the variables: $\frac{1}{y}dy = \tan(t)dt$.
   - Integrate both sides: $\ln|y| = -\ln|\cos(t)| + C$.
   - Simplify using log laws: $\ln|y| = \ln|\sec(t)| + C$.
   - Exponentiate: $y = A\sec(t)$.
   - Apply initial condition $y(0) = 5$: $5 = A\sec(0) \implies A = 5$.

   Answer: $y(t) = 5\sec(t)$

2. A tank contains 50 liters of water with 100 grams of salt. Brine with 2 grams per liter flows in at 3 liters per minute, and the mixture flows out at 3 liters per minute. Find the ODE for the salt $y(t)$ and the equilibrium value.

   - Rate in: 3 L/min $\times$ 2 g/L = 6 g/min.
   - Rate out: 3 L/min $\times \frac{y}{50}$ g/L $= \frac{3y}{50}$ g/min.
   - ODE: $\frac{dy}{dt} = 6 - \frac{3y}{50}$.
   - Equilibrium: Set $\frac{dy}{dt} = 0 \implies 6 = \frac{3y}{50} \implies 300 = 3y \implies y = 100$.

   Answer: $\frac{dy}{dt} = 6 - \frac{3y}{50}, \quad y^* = 100$

3. Consider the autonomous equation $y' = y(y-2)(y-4)$. Identify all equilibria and classify their stability.

   - Equilibria are the roots of $f(y) = y(y-2)(y-4)$, which are $y = 0, 2, 4$.
   - Compute $f'(y) = (y-2)(y-4) + y(y-4) + y(y-2)$.
   - At $y = 0$: $f'(0) = (-2)(-4) = 8 > 0 \implies$ Unstable.
   - At $y = 2$: $f'(2) = 2(2-4) = -4 < 0 \implies$ Stable.
   - At $y = 4$: $f'(4) = 4(4-2) = 8 > 0 \implies$ Unstable.

   Answer: $y = 0$ (Unstable), $y = 2$ (Stable), $y = 4$ (Unstable)
