Stability on the phase line; bifurcation teaser

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Practice problems

1. Find all equilibrium solutions for the autonomous ODE $y' = y^3 - 4y$ and classify each as stable, unstable, or semi-stable.
2. Consider the ODE $y' = y^2 - 3y + 2$. Determine the stability of its equilibria.
3. Analyze the stability of the equilibrium at $y = 0$ for the equation $y' = y^3$. Is it stable, unstable, or semi-stable?

Solutions

1. Find all equilibrium solutions for the autonomous ODE $y' = y^3 - 4y$ and classify each as stable, unstable, or semi-stable.

    - Set $f(y) = y^3 - 4y = 0$. Factoring gives $y(y-2)(y+2) = 0$, so equilibria are $y = -2, 0, 2$.
    - Compute the derivative $f'(y) = 3y^2 - 4$.
    - At $y = -2$, $f'(-2) = 3(4) - 4 = 8 > 0$, so it is unstable.
    - At $y = 0$, $f'(0) = -4 < 0$, so it is stable.
    - At $y = 2$, $f'(2) = 3(4) - 4 = 8 > 0$, so it is unstable.

    Answer: y = -2 (unstable), y = 0 (stable), y = 2 (unstable)

2. Consider the ODE $y' = y^2 - 3y + 2$. Determine the stability of its equilibria.

    - Set $y^2 - 3y + 2 = 0$. Factoring gives $(y-1)(y-2) = 0$, so equilibria are $y = 1$ and $y = 2$.
    - Compute $f'(y) = 2y - 3$.
    - At $y = 1$, $f'(1) = 2(1) - 3 = -1 < 0$, so it is stable.
    - At $y = 2$, $f'(2) = 2(2) - 3 = 1 > 0$, so it is unstable.

    Answer: y = 1 (stable), y = 2 (unstable)

3. Analyze the stability of the equilibrium at $y = 0$ for the equation $y' = y^3$. Is it stable, unstable, or semi-stable?

    - The equilibrium is at $y = 0$ since $f(0) = 0^3 = 0$.
    - Compute $f'(y) = 3y^2$. At $y = 0$, $f'(0) = 0$, so the derivative test is inconclusive.
    - Check the sign of $f(y) = y^3$. For $y > 0$, $f(y) > 0$ (flow is away from 0).
    - For $y < 0$, $f(y) < 0$ (flow is away from 0).
    - Since the flow is away from the equilibrium on both sides, it is unstable.

    Answer: unstable
