Autonomous equations and equilibria — the phase line

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Practice problems

1. Find the equilibrium solutions of $y' = y^3 - y$ and classify each as stable, unstable, or semi-stable.
2. Consider the autonomous equation $y' = y(2 - y)$. Determine the long-term behavior of the solution $y(t)$ if the initial condition is $y(0) = 3$.
3. For what values of the constant k does the equation $y' = y^2 - k$ have exactly one semi-stable equilibrium?

Solutions

1. Find the equilibrium solutions of $y' = y^3 - y$ and classify each as stable, unstable, or semi-stable.

   - Set $y^3 - y = 0$, which factors as $y(y-1)(y+1) = 0$.
   - The equilibria are $y = -1, 0, 1$.
   - For $y > 1$, $y' > 0$ (up). For $0 < y < 1$, $y' < 0$ (down). For $-1 < y < 0$, $y' > 0$ (up). For $y < -1$, $y' < 0$ (down).
   - At $y = 1$, arrows point away (source). At $y = 0$, arrows point toward (sink). At $y = -1$, arrows point away (source).

   Answer: $y = 0$ is stable; $y = 1$ and $y = -1$ are unstable.

2. Consider the autonomous equation $y' = y(2 - y)$. Determine the long-term behavior of the solution $y(t)$ if the initial condition is $y(0) = 3$.

   - Find equilibria: $y(2 - y) = 0$ gives $y = 0$ and $y = 2$.
   - Check the sign for $y > 2$: if $y = 3$, $y' = 3(2 - 3) = -3$, which is negative.
   - Since the derivative is negative for $y > 2$, the solution $y(t)$ will decrease toward the equilibrium at $y = 2$.
   - Because $y = 2$ is a sink (arrows from above point down, arrows from below point up), the solution converges to 2.

   Answer: $y(t) \to 2$ as $t \to \infty$.

3. For what values of the constant k does the equation $y' = y^2 - k$ have exactly one semi-stable equilibrium?

   - Equilibria occur where $y^2 - k = 0$, so $y^2 = k$.
   - If $k > 0$, there are two distinct equilibria $y = \pm\sqrt{k}$, which are typically a sink and a source.
   - If $k < 0$, there are no real equilibria.
   - If $k = 0$, the equation is $y' = y^2$. We found in the lesson that $y = 0$ is a semi-stable equilibrium because $y^2 \geq 0$ for all y.

   Answer: $k = 0$
