Mixing problems — tanks and pollutants

Lesson 14 · Module 1 — First-Order Modelling · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. A 500 L tank initially contains 10 kg of salt. Brine containing 1 kg/L of salt flows in at 5 L/min, and the well-mixed solution flows out at 5 L/min. Find the amount of salt $x(t)$ at any time t.
2. A 200 L tank is filled with pure water. Brine with 0.5 kg/L flows in at 3 L/min and flows out at 3 L/min. How long does it take for the salt amount to reach 50 kg?
3. A 100 L tank contains pure water. Brine with 2 g/L flows in at 5 L/min, but flows out at 3 L/min. Find the amount of salt $x(t)$ in the tank after 10 minutes.

Solutions

1. A 500 L tank initially contains 10 kg of salt. Brine containing 1 kg/L of salt flows in at 5 L/min, and the well-mixed solution flows out at 5 L/min. Find the amount of salt $x(t)$ at any time t.

   - Identify parameters: $V = 500$, $r_{in} = 5$, $c_{in} = 1$, $r_{out} = 5$.
   - Set up the ODE: $x' = (5 \times 1) - (5 \times x/500)$, which simplifies to $x' = 5 - x/100$.
   - Solve using integrating factor $e^{t/100}$: $x(t) = 500 + Ce^{-t/100}$.
   - Apply initial condition $x(0) = 10$: $10 = 500 + C \implies C = -490$.
   - Final solution: $x(t) = 500 - 490e^{-t/100}$.

   Answer: x(t) = 500 - 490eˆ-t/100

2. A 200 L tank is filled with pure water. Brine with 0.5 kg/L flows in at 3 L/min and flows out at 3 L/min. How long does it take for the salt amount to reach 50 kg?

   - Set up ODE: $x' = (3 \times 0.5) - (3 \times x/200) = 1.5 - 0.015x$.
   - Solve the ODE: $x(t) = 100(1 - e^{-0.015t})$.
   - Set $x(t) = 50$: $50 = 100(1 - e^{-0.015t}) \implies 0.5 = 1 - e^{-0.015t}$.
   - Solve for t: $e^{-0.015t} = 0.5 \implies -0.015t = \ln(0.5)$.
   - Calculate t: $t = \ln(0.5)/-0.015 \approx 46.21$ minutes.

   Answer: t \approx 46.21 \text minutes

3. A 100 L tank contains pure water. Brine with 2 g/L flows in at 5 L/min, but flows out at 3 L/min. Find the amount of salt $x(t)$ in the tank after 10 minutes.

   - Volume function: $V(t) = 100 + (5 - 3)t = 100 + 2t$.
   - Set up ODE: $x' = (5 \times 2) - (3 \times x/(100 + 2t)) = 10 - 3x/(100 + 2t)$.
   - Integrating factor: $\int \frac{3}{100+2t} dt = \frac{3}{2} \ln(100 + 2t)$, so $\mu(t) = (100 + 2t)^{1.5}$.
   - Integrate: $x(100+2t)^{1.5} = \int 10(100+2t)^{1.5} dt = \frac{10}{2\times 2.5}(100+2t)^{2.5} + C = 2(100+2t)^{2.5} + C$.
   - General solution: $x(t) = 2(100 + 2t) + C(100 + 2t)^{-1.5}$.
   - Initial condition $x(0) = 0$: $0 = 200 + C(100)^{-1.5} \implies C = -200 \times 1000 = -200,000$.
   - At $t = 10$: $x(10) = 2(120) - 200,000(120)^{-1.5} \approx 240 - 152.1 = 87.9$ g.

   Answer: x(10) \approx 87.9 \text g
