The modelling loop: world -¿ equation -¿ solution -¿ world

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Practice problems

1. A metal sphere is heated to 100 degrees Celsius and then placed in a room with a constant temperature of 20 degrees Celsius. The rate of change of the sphere's temperature T is proportional to the difference between T and the room temperature. Write the ODE and find the particular solution $T(t)$ if the temperature after 10 minutes is 60 degrees Celsius.

2. A population of fish in a pond grows at a rate proportional to the current population P, but fish are harvested at a constant rate of 500 fish per year. If the pond starts with 2000 fish and the growth constant is $k = 0.1$ per year, find the equation for $P(t)$ and determine if the population will eventually go extinct.

3. A tank contains 500 liters of brine with 20 kg of salt. Brine containing 0.1 kg of salt per liter flows in at 5 L/min. The mixture flows out at 5 L/min. Find the amount of salt $y(t)$ in the tank at any time t and find the steady state amount of salt as $t \to \infty$.

Solutions

1. A metal sphere is heated to 100 degrees Celsius and then placed in a room with a constant temperature of 20 degrees Celsius. The rate of change of the sphere's temperature T is proportional to the difference between T and the room temperature. Write the ODE and find the particular solution $T(t)$ if the temperature after 10 minutes is 60 degrees Celsius.

   - Step 1: Translate the rule to an ODE. The rate of change is dT/dt. The difference is $(T-20)$. Since it is cooling, the rate is negative: $dT/dt = -k(T-20)$.
   - Step 2: Solve the separable ODE. $\int \frac{1}{T-20} dT = \int -k dt$, which gives $\ln(T-20) = -kt + C$, or $T(t) = 20 + Ce^{-kt}$.
   - Step 3: Use initial condition $T(0) = 100$. $100 = 20 + Ce^0$, so $C = 80$. The solution is $T(t) = 20 + 80e^{-kt}$.
   - Step 4: Use $T(10) = 60$ to find k. $60 = 20 + 80e^{-10k} \implies 40 = 80e^{-10k} \implies 0.5 = e^{-10k}$. Thus $-10k = \ln(0.5)$, so $k = \frac{\ln(2)}{10} \approx 0.0693$.

   Answer: $T(t) = 20 + 80e^{-0.0693t}$

2. A population of fish in a pond grows at a rate proportional to the current population P, but fish are harvested at a constant rate of 500 fish per year. If the pond starts with 2000 fish and the growth constant is $k = 0.1$ per year, find the equation for $P(t)$ and determine if the population will eventually go extinct.

   - Step 1: Write the ODE. Growth is $0.1P$ and harvest is -500. So $dP/dt = 0.1P - 500$.
   - Step 2: Solve the linear ODE. This is separable: $dP/(0.1P - 500) = dt$. Integrating gives $10\ln(0.1P - 500) = t + C$, or $0.1P - 500 = Ae^{0.1t}$.
   - Step 3: Use $P(0) = 2000$. $0.1(2000) - 500 = Ae^0 \implies 200 - 500 = A \implies A = -300$.
   - Step 4: Solve for P. $0.1P = 500 - 300e^{0.1t} \implies P(t) = 5000 - 3000e^{0.1t}$.
   - Step 5: Analyze long term behavior. As $t \to \infty$, $e^{0.1t} \to \infty$, so $P(t) \to -\infty$. The population will hit zero and go extinct.

   Answer: $P(t) = 5000 - 3000e^{0.1t}$; Yes, it goes extinct.

3. A tank contains 500 liters of brine with 20 kg of salt. Brine containing 0.1 kg of salt per liter flows in at 5 L/min. The mixture flows out at 5 L/min. Find the amount of salt $y(t)$ in the tank at any time t and find the steady state amount of salt as $t \to \infty$.

   - Step 1: Write the ODE. Rate in is 0.1 kg/L $\times$ 5 L/min $=$ 0.5 kg/min. Rate out is $(y/500)$ kg/L $\times$ 5 L/min $= 0.01y$ kg/min.
   - Step 2: The ODE is $dy/dt = 0.5 - 0.01y$.
   - Step 3: Solve the ODE. $\int \frac{1}{0.5 - 0.01y} dy = \int dt \implies -100\ln(0.5 - 0.01y) = t + C$.
   - Step 4: $0.5 - 0.01y = Ae^{-0.01t} \implies 0.01y = 0.5 - Ae^{-0.01t} \implies y(t) = 50 - 100Ae^{-0.01t}$.
   - Step 5: Use $y(0) = 20$. $20 = 50 - 100A \implies 100A = 30$. So $y(t) = 50 - 30e^{-0.01t}$.
   - Step 6: Steady state is $\lim_{t\to\infty}(50 - 30e^{-0.01t}) = 50$ kg.

   Answer: $y(t) = 50 - 30e^{-0.01t}$; Steady state is 50 kg.