Module 0 review + self-test

Lesson 10 · Module 0 — What an ODE Is · Differential Equations · dailymaths.uk/diffeq

Practice problems

1. Solve the initial value problem $y' + 2y = 4$ with $y(0) = 3$.
2. Solve the differential equation $y' = \frac{y^2}{t^2}$.
3. Solve the Bernoulli equation $y' + \frac{1}{t}y = ty^2$ for $t > 0$.

Solutions

1. Solve the initial value problem $y' + 2y = 4$ with $y(0) = 3$.

   - Identify the equation as first order linear with $P(t) = 2$ and $Q(t) = 4$.
   - Compute the integrating factor $\mu(t) = e^{\int 2dt} = e^{2t}$.
   - Multiply the ODE by $\mu(t)$ to get $(e^{2t}y)' = 4e^{2t}$.
   - Integrate both sides: $e^{2t}y = 2e^{2t} + C$.
   - Solve for y: $y = 2 + Ce^{-2t}$.
   - Apply $y(0) = 3$: $3 = 2 + C \implies C = 1$.

   Answer: $y(t) = 2 + e^{-2t}$

2. Solve the differential equation $y' = \frac{y^2}{t^2}$.

   - Identify the equation as homogeneous since the right side depends only on y/t.
   - Substitute $v = y/t$, so $y = vt$ and $y' = v + tv'$.
   - Rewrite the ODE: $v + tv' = v^2$.
   - Separate variables: $tv' = v^2 - v \implies \frac{dv}{v(v-1)} = \frac{dt}{t}$.
   - Integrate using partial fractions: $\ln|v-1| - \ln|v| = \ln|t| + C$.
   - Simplify: $\frac{v-1}{v} = Ct \implies 1 - \frac{1}{v} = Ct$.
   - Substitute back $v = y/t$: $1 - \frac{t}{y} = Ct$.

   Answer: $y(t) = \frac{t}{1-Ct}$

3. Solve the Bernoulli equation $y' + \frac{1}{t}y = ty^2$ for $t > 0$.

   - Identify as Bernoulli with $n = 2$.
   - Substitute $v = y^{1-2} = y^{-1}$, so $v' = -y^{-2}y'$.
   - Divide the original ODE by y^2: $y^{-2}y' + \frac{1}{t}y^{-1} = t$.
   - Substitute v: $-v' + \frac{1}{t}v = t \implies v' - \frac{1}{t}v = -t$.
   - Find integrating factor $\mu(t) = e^{\int -1/tdt} = e^{-\ln t} = 1/t$.
   - Multiply: $(v/t)' = -1$.
   - Integrate: $v/t = -t + C \implies v = -t^2 + Ct$.
   - Substitute back $y = 1/v$.

   Answer: $y(t) = \frac{1}{Ct-t^2}$