Substitution methods: Bernoulli and homogeneous equations

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Practice problems

1. Solve the Bernoulli equation $y' + \frac{1}{t}y = ty^3$ for $t > 0$.
2. Solve the homogeneous equation $y' = \frac{y^2+t^2}{ty}$.
3. Solve $y' = \frac{y}{t} + t^2y^2$ with $y(1) = 1$.

Solutions

1. Solve the Bernoulli equation $y' + \frac{1}{t}y = ty^3$ for $t > 0$.

   - Identify $n = 3$, so use substitution $v = y^{1-3} = y^{-2}$.
   - Compute $v' = -2y^{-3}y'$.
   - Multiply the ODE by $-2y^{-3}$ to get $-2y^{-3}y' - \frac{2}{t}y^{-2} = -2t$.
   - Substitute v and v' to get $v' - \frac{2}{t}v = -2t$.
   - Use integrating factor $\mu(t) = e^{\int -2/tdt} = t^{-2}$.
   - Solve $t^{-2}v = \int -2t(t^{-2})dt = -2\ln|t| + C$.
   - Find $v = t^2(-2\ln|t| + C)$ and substitute back $y = \pm 1/\sqrt{v}$.

   Answer: $y = \pm\frac{1}{\sqrt{t^2(C-2\ln|t|)}}$

2. Solve the homogeneous equation $y' = \frac{y^2+t^2}{ty}$.

   - Rewrite as $y' = \frac{y}{t} + \frac{t}{y}$, which is $f(v) = v + 1/v$ where $v = y/t$.
   - Substitute $y' = v + tv'$, so $v + tv' = v + 1/v$.
   - Simplify to $tv' = 1/v$.
   - Separate variables: $vdv = \frac{1}{t}dt$.
   - Integrate: $\frac{1}{2}v^2 = \ln|t| + C$.
   - Substitute $v = y/t$: $\frac{y^2}{2t^2} = \ln|t| + C$.

   Answer: $y^2 = 2t^2(\ln|t| + C)$

3. Solve $y' = \frac{y}{t} + t^2y^2$ with $y(1) = 1$.

   - This is a Bernoulli equation with $n = 2$. Use $v = y^{1-2} = y^{-1}$.
   - Then $v' = -y^{-2}y'$.
   - Multiply ODE by $-y^{-2}$: $-y^{-2}y' = -y^{-2}(y/t) - t^2$.
   - Substitute: $v' = -v/t - t^2$, or $v' + \frac{1}{t}v = -t^2$.
   - Integrating factor $\mu(t) = e^{\int 1/tdt} = t$.
   - Solve $tv = \int -t^3dt = -\frac{1}{4}t^4 + C$.
   - Find $v = -\frac{1}{4}t^3 + C/t$.
   - Substitute $y = 1/v$: $y = \frac{1}{C/t - t^3/4}$.
   - Use $y(1) = 1$: $1 = \frac{1}{C-1/4} \implies C = 5/4$.

   Answer: $y = \frac{4t}{5-t^4}$
