First-order linear equations and integrating factors

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Practice problems

1. Solve the initial value problem $y' + 2y = e^{-t}$ with $y(0) = 1$.
2. Find the general solution for $y' - \frac{1}{t}y = t^2$ for $t > 0$.
3. Solve $y' + ty = te^{-t^2/2}$.

Solutions

1. Solve the initial value problem $y' + 2y = e^{-t}$ with $y(0) = 1$.

    - Identify $P(t) = 2$ and $Q(t) = e^{-t}$.
    - Compute the integrating factor $\mu(t) = e^{\int 2dt} = e^{2t}$.
    - Multiply the ODE by e^{2t} to get $\frac{d}{dt}(e^{2t}y) = e^{2t}e^{-t} = e^{t}$.
    - Integrate both sides: $e^{2t}y = e^{t} + C$.
    - Solve for y: $y = e^{-t} + Ce^{-2t}$.
    - Use $y(0) = 1$: $1 = 1 + C \implies C = 0$.
    - The particular solution is $y = e^{-t}$.

    Answer: $y = e^{-t}$

2. Find the general solution for $y' - \frac{1}{t}y = t^2$ for $t > 0$.

    - Identify $P(t) = -1/t$ and $Q(t) = t^2$.
    - Compute $\mu(t) = e^{\int -1/t dt} = e^{-\ln t} = e^{\ln(1/t)} = 1/t$.
    - Multiply the ODE by $1/t$: $\frac{d}{dt}(\frac{1}{t}y) = \frac{1}{t}t^2 = t$.
    - Integrate both sides: $\frac{1}{t}y = \frac{1}{2}t^2 + C$.
    - Solve for y: $y = \frac{1}{2}t^3 + Ct$.

    Answer: $y = \frac{1}{2}t^3 + Ct$

3. Solve $y' + ty = te^{-t^2/2}$.

    - Identify $P(t) = t$ and $Q(t) = te^{-t^2/2}$.
    - Compute $\mu(t) = e^{\int tdt} = e^{t^2/2}$.
    - Multiply the ODE by $e^{t^2/2}$: $\frac{d}{dt}(e^{t^2/2}y) = e^{t^2/2}te^{-t^2/2} = t$.
    - Integrate both sides: $e^{t^2/2}y = \frac{1}{2}t^2 + C$.
    - Solve for y: $y = (\frac{1}{2}t^2 + C)e^{-t^2/2}$.

    Answer: $y = (\frac{1}{2}t^2 + C)e^{-t^2/2}$
