Existence and uniqueness — when is the future determined?

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Practice problems

1. Does the initial value problem $y' = y^2 + t$ with $y(0) = 0$ have a guaranteed unique solution near the origin? Explain why.
2. Consider $y' = 2\sqrt{|y|}$ with $y(0) = 0$. Show that $y(t) = 0$ is a solution and check if the uniqueness condition is satisfied at the origin.
3. For the ODE $y' = \frac{1}{t-1}y$, does a unique solution exist for the initial condition $y(1) = 2$? Why or why not?

Solutions

1. Does the initial value problem $y' = y^2 + t$ with $y(0) = 0$ have a guaranteed unique solution near the origin? Explain why.

   - Identify $f(t, y) = y^2 + t$. This is a polynomial, so it is continuous everywhere, including $(0, 0)$.
   - Compute the partial derivative with respect to y: $\frac{\partial f}{\partial y} = 2y$.
   - Check continuity of the partial derivative: $2y$ is a polynomial and is continuous everywhere, including $(0, 0)$.
   - Since both f and $\frac{\partial f}{\partial y}$ are continuous, a unique solution is guaranteed.

   Answer: Yes, because both $f(t, y) = y^2 + t$ and $\frac{\partial f}{\partial y} = 2y$ are continuous at $(0, 0)$.

2. Consider $y' = 2\sqrt{|y|}$ with $y(0) = 0$. Show that $y(t) = 0$ is a solution and check if the uniqueness condition is satisfied at the origin.

   - Substitute $y(t) = 0$ into the ODE: $y' = 0$ and $2\sqrt{|0|} = 0$. Since $0 = 0$, it is a solution.
   - Identify $f(t, y) = 2\sqrt{|y|}$.
   - Compute the partial derivative: $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{|y|}}$ (for $y \neq 0$).
   - Evaluate at the initial point $y = 0$: the expression $\frac{1}{\sqrt{0}}$ is undefined (approaches infinity).
   - The uniqueness condition is not satisfied.

   Answer: The uniqueness condition is not satisfied because $\frac{\partial f}{\partial y}$ is undefined at $y = 0$.

3. For the ODE $y' = \frac{1}{t-1}y$, does a unique solution exist for the initial condition $y(1) = 2$? Why or why not?

   - Identify $f(t, y) = \frac{y}{t-1}$.
   - Check continuity of f at the point $(1, 2)$.
   - The denominator $t - 1$ becomes $1 - 1 = 0$ at $t = 1$.
   - Since the function f is undefined at $t = 1$, it is not continuous at the initial point.
   - Therefore, the existence of a solution is not guaranteed by the Picard theorem.

   Answer: No, because $f(t, y)$ is not continuous at $t = 1$.
