Slope fields — seeing every solution at once

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Practice problems

1. Find the equilibrium solutions for the differential equation $y' = y(y-3)$ and determine if they are stable (attractors) or unstable (repellers) by considering the sign of y' between them.

2. Given the slope field for $y' = t - y$, sketch the behavior of a solution that starts at the initial condition $y(0) = 2$. Does the solution approach a specific line as t increases?

3. Determine if the differential equation $y' = \frac{t^2}{y}$ is autonomous. If not, find the slope of the solution curve at the point $(2, 1)$ and describe the concavity of the solution at that point using the formula $y'' = \frac{d}{dt} f(t, y)$.

Solutions

1. Find the equilibrium solutions for the differential equation $y' = y(y-3)$ and determine if they are stable (attractors) or unstable (repellers) by considering the sign of y' between them.

   - Set $y' = 0$, which gives $y(y-3) = 0$.
   - The equilibrium solutions are $y = 0$ and $y = 3$.
   - For $y > 3$, y' is positive, so solutions move away from $y = 3$.
   - For $0 < y < 3$, y' is negative, so solutions move away from $y = 3$ and toward $y = 0$.
   - For $y < 0$, y' is positive, so solutions move toward $y = 0$.
   - Therefore, $y = 3$ is unstable and $y = 0$ is stable.

   Answer: $y = 0$ (stable), $y = 3$ (unstable)

2. Given the slope field for $y' = t - y$, sketch the behavior of a solution that starts at the initial condition $y(0) = 2$. Does the solution approach a specific line as t increases?

   - At $(0, 2)$, the slope is $0 - 2 = -2$, so the curve starts by decreasing.
   - As t increases and y decreases, the slope $t - y$ becomes less negative.
   - We check the line $y = t - 1$. Its slope is 1.
   - Plugging $y = t - 1$ into the ODE gives $t - (t - 1) = 1$.
   - Since the slopes match, $y = t - 1$ is a solution.
   - The solution starting at $(0, 2)$ will be attracted to the line $y = t - 1$.

   Answer: The solution decreases and asymptotically approaches the line $y = t - 1$.

3. Determine if the differential equation $y' = \frac{t^2}{y}$ is autonomous. If not, find the slope of the solution curve at the point $(2, 1)$ and describe the concavity of the solution at that point using the formula $y'' = \frac{d}{dt} f(t, y)$.

   - The equation is not autonomous because the slope depends on t.
   - At $(2, 1)$, the slope is $y' = \frac{2^2}{1} = 4$.
   - To find concavity, we differentiate $y' = t^2 y^{-1}$ with respect to t using the product and chain rules: $y'' = 2ty^{-1} + t^2(-y^{-2}y')$.
   - Substitute $t = 2, y = 1, y' = 4$: $y'' = 2(2)(1)^{-1} + (2^2)(-(1)^{-2})(4)$.
   - Compute: $y'' = 4 - 16 = -12$.
   - Since $y'' < 0$, the solution is concave down at this point.

   Answer: Not autonomous; slope is 4; concave down.