

The Central Limit Theorem II — why the normal appears everywhere

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Practice problems

1. A set of 20 independent random variables each have a mean $\mu = 5$ and a variance $\sigma^2 = 4$. Use the Central Limit Theorem to find the probability that their sum S is greater than 120.
2. A population has a mean of 50 and a standard deviation of 10. If the population is highly skewed, would a sample size of $n = 5$ be sufficient to use the normal approximation for the sample mean? Explain why or why not.
3. Consider a random variable X with a probability density function $f(x) = 1/(\pi(1 + x^2))$ for all real x . If we sum 100 independent copies of X , will the resulting distribution be approximately normal? Why?

Solutions

1. A set of 20 independent random variables each have a mean $\mu = 5$ and a variance $\sigma^2 = 4$. Use the Central Limit Theorem to find the probability that their sum S is greater than 120.
 - Calculate the mean of the sum: $E[S] = 20 \times 5 = 100$.
 - Calculate the variance of the sum: $\text{Var}(S) = 20 \times 4 = 80$.
 - Calculate the standard deviation of the sum: $\sigma_S = \sqrt{80} \approx 8.9443$.
 - Find the z-score for 120: $z = (120 - 100)/8.9443 \approx 2.236$.
 - Find the probability $P(Z > 2.236)$ using the standard normal table: $1 - 0.9873 = 0.0127$.

Answer: 0.0127

2. A population has a mean of 50 and a standard deviation of 10. If the population is highly skewed, would a sample size of $n = 5$ be sufficient to use the normal approximation for the sample mean? Explain why or why not.
 - The Central Limit Theorem states that the sampling distribution approaches normality as n increases.
 - For symmetric populations, this happens quickly (small n).
 - For highly skewed populations, a larger n (typically $n \geq 30$) is required to overcome the skewness.
 - Since $n = 5$ is very small and the population is highly skewed, the approximation would be poor.

Answer: No, because high skewness requires a larger sample size (typically $n \geq 30$) for the sampling distribution to become approximately normal.

3. Consider a random variable X with a probability density function $f(x) = 1/(\pi(1 + x^2))$ for all real x . If we sum 100 independent copies of X , will the resulting distribution be approximately normal? Why?
 - The given PDF is that of a Cauchy distribution.
 - The integral for the variance $\int x^2 f(x) dx$ diverges, meaning the variance is infinite.
 - The Central Limit Theorem requires the population to have a finite variance $\sigma^2 < \infty$.
 - Since the variance is infinite, the sum will not converge to a normal distribution regardless of the sample size.

Answer: No, because the distribution has infinite variance, which violates the primary requirement of the Central Limit Theorem.