The Law of Large Numbers — why casinos always win

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Practice problems

1. A population has a mean μ and a standard deviation $\sigma = 0.5$. Use Chebyshev's inequality to find the minimum sample size n such that the probability that the sample mean $\bar{X}_n$ differs from μ by more than 0.1 is less than 0.05.

2. A game has an expected value of $E[X] = -2$ dollars with a standard deviation of $\sigma = 10$ dollars. If a casino hosts $n = 10,000$ games, what is the maximum probability that the average loss per game is actually a profit for the player (i.e., $\bar{X}_n \geq 0$)?

3. Suppose you have a sequence of independent and identically distributed random variables X_i with mean μ and variance σ^2. Prove that for any $\epsilon > 0$, the probability $P(|\bar{X}_n - \mu| < \epsilon)$ approaches 1 as $n \to \infty$.

Solutions

1. A population has a mean μ and a standard deviation $\sigma = 0.5$. Use Chebyshev's inequality to find the minimum sample size n such that the probability that the sample mean $\bar{X}_n$ differs from μ by more than 0.1 is less than 0.05.

   - We use the bound $P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$.
   - Substitute the given values: $\epsilon = 0.1$ and $\sigma = 0.5$.
   - Set the inequality: $\frac{0.5^2}{n \cdot 0.1^2} < 0.05$.
   - Simplify the constants: $\frac{0.25}{n \cdot 0.01} < 0.05$, which becomes $\frac{25}{n} < 0.05$.
   - Solve for n: $n > \frac{25}{0.05} = 500$.

   Answer: n ¿ 500

2. A game has an expected value of $E[X] = -2$ dollars with a standard deviation of $\sigma = 10$ dollars. If a casino hosts $n = 10,000$ games, what is the maximum probability that the average loss per game is actually a profit for the player (i.e., $\bar{X}_n \geq 0$)?

   - The deviation from the mean $\mu = -2$ to the threshold 0 is $\epsilon = 2$.
   - The variance of the sample mean is $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n} = \frac{10^2}{10000} = \frac{100}{10000} = 0.01$.
   - Apply Chebyshev's inequality: $P(|\bar{X}_n - (-2)| \geq 2) \leq \frac{0.01}{2^2}$.
   - Calculate the result: $\frac{0.01}{4} = 0.0025$.

   Answer: 0.0025

3. Suppose you have a sequence of independent and identically distributed random variables X_i with mean μ and variance σ^2. Prove that for any $\epsilon > 0$, the probability $P(|\bar{X}_n - \mu| < \epsilon)$ approaches 1 as $n \to \infty$.

   - Start with the complement: $P(|\bar{X}_n - \mu| < \epsilon) = 1 - P(|\bar{X}_n - \mu| \geq \epsilon)$.
   - Use the Law of Large Numbers bound: $P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$.
   - Therefore, $P(|\bar{X}_n - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2}$.
   - Take the limit as $n \to \infty$: $\lim_{n\to\infty}(1 - \frac{\sigma^2}{n\epsilon^2}) = 1 - 0 = 1$.
   - Since the probability is bounded above by 1 and below by a value approaching 1, it must converge to 1.

   Answer: The probability converges to 1.