

Module 4 Review + Self-Test

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Practice problems

1. Given the joint density $f_{X,Y}(x,y) = 8xy$ for $0 < y < x < 1$ and 0 otherwise, find the marginal density $f_X(x)$ and the marginal density $f_Y(y)$.
2. For the joint density $f_{X,Y}(x,y) = 8xy$ for $0 < y < x < 1$, compute the covariance $\text{Cov}(X, Y)$.
3. Suppose X is a random variable and $Y|X = x$ follows a normal distribution with mean x and variance 1. If X itself follows a normal distribution with mean 0 and variance 1, use the Law of Iterated Expectations to find $E[Y]$.

Solutions

1. Given the joint density $f_{X,Y}(x, y) = 8xy$ for $0 < y < x < 1$ and 0 otherwise, find the marginal density $f_X(x)$ and the marginal density $f_Y(y)$.

- To find $f_X(x)$, integrate $8xy$ with respect to y from 0 to x .
- $\int_0^x 8xy dy = [4xy^2]_0^x = 4x^3$.
- To find $f_Y(y)$, integrate $8xy$ with respect to x from y to 1.
- $\int_y^1 8xy dx = [4x^2y]_y^1 = 4y(1 - y^2) = 4y - 4y^3$.

Answer: $f_X(x) = 4x^3$, $f_Y(y) = 4y - 4y^3$

2. For the joint density $f_{X,Y}(x, y) = 8xy$ for $0 < y < x < 1$, compute the covariance $\text{Cov}(X, Y)$.

- Compute $E[X] = \int_0^1 x(4x^3) dx = [4x^5/5]_0^1 = 0.8$.
- Compute $E[Y] = \int_0^1 y(4y - 4y^3) dy = [4y^3/3 - 4y^5/5]_0^1 = 4/3 - 4/5 = 8/15 \approx 0.533$.
- Compute $E[XY] = \int_0^1 \int_0^x xy(8xy) dy dx = \int_0^1 [8x^2y^3/3]_0^x dx = \int_0^1 8x^5/3 dx = [8x^6/18]_0^1 = 4/9 \approx 0.444$.
- $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 4/9 - (4/5)(8/15) = 4/9 - 32/75 = (100 - 96)/225 = 4/225$.

Answer: $\text{Cov}(X, Y) = 4/225$

3. Suppose X is a random variable and $Y|X = x$ follows a normal distribution with mean x and variance 1. If X itself follows a normal distribution with mean 0 and variance 1, use the Law of Iterated Expectations to find $E[Y]$.

- We are given the conditional expectation $E[Y|X = x] = x$.
- Therefore, the conditional expectation as a random variable is $E[Y|X] = X$.
- By the Law of Iterated Expectations, $E[Y] = E[E[Y|X]]$.
- Substituting the conditional expectation, we get $E[Y] = E[X]$.
- Since X has a mean of 0, $E[Y] = 0$.

Answer: $E[Y] = 0$