

Conditional expectation — the best guess

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Practice problems

1. A joint PMF is given by $P(X = 1, Y = 1) = 0.2$, $P(X = 1, Y = 2) = 0.3$, $P(X = 2, Y = 1) = 0.1$, and $P(X = 2, Y = 2) = 0.4$. Compute $E[Y \mid X = 1]$.
2. Given the joint density $f_{X,Y}(x, y) = 2$ for $0 < x < 1$ and $0 < y < x$, and 0 otherwise, find the function $g(x) = E[Y \mid X = x]$.
3. If X and Y are independent random variables, prove that $E[Y \mid X] = E[Y]$.

Solutions

1. A joint PMF is given by $P(X = 1, Y = 1) = 0.2$, $P(X = 1, Y = 2) = 0.3$, $P(X = 2, Y = 1) = 0.1$, and $P(X = 2, Y = 2) = 0.4$. Compute $E[Y | X = 1]$.
 - Find the marginal $P(X = 1) = 0.2 + 0.3 = 0.5$.
 - Find conditional probabilities: $P(Y = 1 | X = 1) = 0.2/0.5 = 0.4$ and $P(Y = 2 | X = 1) = 0.3/0.5 = 0.6$.
 - Compute expectation: $E[Y | X = 1] = 1(0.4) + 2(0.6) = 0.4 + 1.2 = 1.6$.

Answer: 1.6

2. Given the joint density $f_{X,Y}(x, y) = 2$ for $0 < x < 1$ and $0 < y < x$, and 0 otherwise, find the function $g(x) = E[Y | X = x]$.
 - Find marginal $f_X(x) = \int_0^x 2 \, dy = 2x$.
 - Find conditional density $f_{Y|X}(y | x) = 2/(2x) = 1/x$ for $0 < y < x$.
 - Compute expectation: $E[Y | X = x] = \int_0^x y(1/x) \, dy = [y^2/(2x)]_0^x = x^2/(2x) = x/2$.

Answer: $x/2$

3. If X and Y are independent random variables, prove that $E[Y | X] = E[Y]$.
 - By independence, the conditional density $f_{Y|X}(y | x) = f_Y(y)$.
 - Substitute this into the conditional expectation formula: $E[Y | X = x] = \int y f_Y(y) \, dy$.
 - The resulting integral is the definition of the marginal expectation $E[Y]$.

Answer: $E[Y]$