

The bivariate normal — the 3D bell

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Practice problems

1. Suppose X and Y follow a bivariate normal distribution with $\mu_X = 10$, $\mu_Y = 50$, $\sigma_X = 2$, $\sigma_Y = 10$, and $\rho = -0.4$. Find the conditional expectation $E[Y \mid X = 12]$.
2. For the same distribution as Problem 1, calculate the conditional variance $\text{Var}(Y \mid X = 12)$.
3. If two variables X and Y are bivariate normal with $\rho = 1$, what is the relationship between z_X and z_Y ? What does this imply about the shape of the 3D probability surface?

Solutions

1. Suppose X and Y follow a bivariate normal distribution with $\mu_X = 10$, $\mu_Y = 50$, $\sigma_X = 2$, $\sigma_Y = 10$, and $\rho = -0.4$. Find the conditional expectation $E[Y \mid X = 12]$.

- First, calculate the z-score of X : $z_X = (12 - 10)/2 = 1$.
- Use the conditional mean formula: $E[Y \mid X = x] = \mu_Y + \rho\sigma_Y z_X$.
- Substitute the values: $50 + (-0.4) \cdot 10 \cdot 1$.
- Calculate the result: $50 - 4 = 46$.

Answer: 46

2. For the same distribution as Problem 1, calculate the conditional variance $\text{Var}(Y \mid X = 12)$.

- Use the conditional variance formula: $\text{Var}(Y \mid X) = \sigma_Y^2(1 - \rho^2)$.
- Substitute the values: $10^2 \cdot (1 - (-0.4)^2)$.
- Calculate the square: $100 \cdot (1 - 0.16)$.
- Final result: $100 \cdot 0.84 = 84$.

Answer: 84

3. If two variables X and Y are bivariate normal with $\rho = 1$, what is the relationship between z_X and z_Y ? What does this imply about the shape of the 3D probability surface?

- If $\rho = 1$, the conditional mean formula gives $E[z_Y \mid z_X] = 1 \cdot z_X$, so $z_Y = z_X$.
- The conditional variance becomes $\sigma_Y^2(1 - 1^2) = 0$.
- This means for any given X , there is only one possible value for Y .
- The 3D surface collapses from a bell into a 2D plane (a ridge) slicing through the 3D space.

Answer: $z_X = z_Y$; the surface collapses into a 2D ridge.