

The Gamma and Beta Distributions

Lesson 52 · Module 3 — Continuous Random Variables · Probability & Statistics · daily-maths.uk/probstat

Practice problems

1. Let X be a Gamma random variable with shape $\alpha = 2$ and rate $\beta = 0.5$. Calculate $P(X < 1)$.
2. A random variable X follows a Beta distribution with $\alpha = 3$ and $\beta = 2$. Find the probability $P(X > 0.5)$.
3. If X_1, X_2, X_3 are independent exponential random variables with rate $\lambda = 4$, what is the PDF of $Y = X_1 + X_2 + X_3$?

Solutions

1. Let X be a Gamma random variable with shape $\alpha = 2$ and rate $\beta = 0.5$. Calculate $P(X < 1)$.

- The PDF is $f(x) = \frac{0.5^2}{\Gamma(2)} x^{2-1} e^{-0.5x} = 0.25xe^{-0.5x}$.
- Integrate $\int_0^1 0.25xe^{-0.5x} dx$ using integration by parts.
- Let $u = 0.25x$ and $dv = e^{-0.5x} dx$, so $du = 0.25dx$ and $v = -2e^{-0.5x}$.
- The integral is $[-0.5xe^{-0.5x}]_0^1 - \int_0^1 -0.5e^{-0.5x} dx$.
- This evaluates to $-0.5e^{-0.5} + [-e^{-0.5x}]_0^1 = -0.5e^{-0.5} - e^{-0.5} + 1 = 1 - 1.5e^{-0.5}$.

Answer: $1 - 1.5 e^{-0.5} \approx 0.0902$

2. A random variable X follows a Beta distribution with $\alpha = 3$ and $\beta = 2$. Find the probability $P(X > 0.5)$.

- First find $B(3, 2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{2 \cdot 1}{24} = \frac{1}{12}$.
- The PDF is $f(x) = 12x^{3-1}(1-x)^{2-1} = 12x^2(1-x) = 12x^2 - 12x^3$.
- Integrate from 0.5 to 1: $\int_{0.5}^1 (12x^2 - 12x^3) dx$.
- The antiderivative is $4x^3 - 3x^4$.
- Evaluate at boundaries: $(4(1)^3 - 3(1)^4) - (4(0.5)^3 - 3(0.5)^4) = 1 - (0.5 - 0.1875) = 1 - 0.3125 = 0.6875$.

Answer: 0.6875

3. If X_1, X_2, X_3 are independent exponential random variables with rate $\lambda = 4$, what is the PDF of $Y = X_1 + X_2 + X_3$?

- The sum of k independent exponential variables with rate β follows a Gamma distribution with shape $\alpha = k$ and rate β .
- Here $k = 3$ and $\beta = 4$.
- The PDF is $f(y) = \frac{4^3}{\Gamma(3)} y^{3-1} e^{-4y}$.
- Since $\Gamma(3) = 2! = 2$, the constant is $64/2 = 32$.
- The final PDF is $f(y) = 32y^2 e^{-4y}$ for $y > 0$.

Answer: $f(y) = 32 y^2 e^{-4y}$