

The normal distribution II — why root two pi?

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Practice problems

1. Given a normal distribution with $\mu = 5$ and $\sigma = \sqrt{2}$, calculate the exact value of the normalization constant.
2. If the standard deviation of a normal distribution is doubled, how does the maximum height of the PDF change?
3. Show that for any μ and σ , the integral $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$ using the substitution $u = \frac{x-\mu}{\sigma\sqrt{2}}$.

Solutions

1. Given a normal distribution with $\mu = 5$ and $\sigma = \sqrt{2}$, calculate the exact value of the normalization constant.

- The normalization constant is given by the formula $\frac{1}{\sigma\sqrt{2\pi}}$
- Substitute $\sigma = \sqrt{2}$ into the formula: $\frac{1}{\sqrt{2}\sqrt{2\pi}}$
- Simplify the denominator: $\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{\pi} = 2\sqrt{\pi}$
- The constant is $\frac{1}{2\sqrt{\pi}}$

Answer: $\frac{1}{2\sqrt{\pi}}$

2. If the standard deviation of a normal distribution is doubled, how does the maximum height of the PDF change?

- The maximum height of the PDF occurs at $x = \mu$ and is equal to the constant $\frac{1}{\sigma\sqrt{2\pi}}$
- Let the original height be $H_1 = \frac{1}{\sigma\sqrt{2\pi}}$
- Let the new standard deviation be $\sigma' = 2\sigma$
- The new height is $H_2 = \frac{1}{2\sigma\sqrt{2\pi}} = \frac{1}{2}H_1$

Answer: The maximum height is halved.

3. Show that for any μ and σ , the integral $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$ using the substitution $u = \frac{x-\mu}{\sigma\sqrt{2}}$.

- Start with the integral $I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$
- Apply substitution $u = \frac{x-\mu}{\sigma\sqrt{2}}$, which means $dx = \sigma\sqrt{2} du$
- Substitute into the integral: $I = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2} (\sigma\sqrt{2} du)$
- Cancel σ and simplify the constants: $I = \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$
- Using the Gaussian integral result $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$, we get $I = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$

Answer: The integral equals 1.