From bars to curves: probability density

Lesson 41 · Module 3 — Continuous Random Variables · Probability & Statistics · daily-maths.uk/probstat

Practice problems

1. Given the function $f(x) = \frac{3}{8}x^2$ for $0 \le x \le 2$ and 0 otherwise, show that it is a valid probability density function and find $P(1 \le X \le 2)$.
2. A continuous random variable has the PDF $f(x) = c(4x - 2x^2)$ for $0 \le x \le 2$ and 0 otherwise. Find the value of the constant c that makes this a valid PDF.
3. For a random variable with PDF $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$, find the probability $P(0 \le X \le 1)$.

Solutions

1. Given the function $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$ and 0 otherwise, show that it is a valid probability density function and find $P(1 \leq X \leq 2)$.

   - Check non-negativity: x^2 is always non-negative, so $f(x) \geq 0$.
   - Check total area: $\int_0^2 \frac{3}{8}x^2\,dx = [\frac{1}{8}x^3]_0^2 = \frac{8}{8} - 0 = 1$.
   - Compute probability: $P(1 \leq X \leq 2) = \int_1^2 \frac{3}{8}x^2\,dx = [\frac{1}{8}x^3]_1^2 = \frac{8}{8} - \frac{1}{8} = \frac{7}{8}$.

   Answer: 0.875

2. A continuous random variable has the PDF $f(x) = c(4x - 2x^2)$ for $0 \leq x \leq 2$ and 0 otherwise. Find the value of the constant c that makes this a valid PDF.

   - Set the total integral to 1: $\int_0^2 c(4x - 2x^2)\,dx = 1$.
   - Integrate: $c[2x^2 - \frac{2}{3}x^3]_0^2 = 1$.
   - Evaluate: $c(2(4) - \frac{2}{3}(8)) = c(8 - \frac{16}{3}) = c(\frac{24-16}{3}) = c\frac{8}{3}$.
   - Solve for c: $c\frac{8}{3} = 1 \implies c = \frac{3}{8}$.

   Answer: 3/8

3. For a random variable with PDF $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$, find the probability $P(0 \leq X \leq 1)$.

   - Set up the integral: $P(0 \leq X \leq 1) = \int_0^1 \frac{1}{\pi(1+x^2)}\,dx$.
   - Recognize the antiderivative: $\int \frac{1}{1+x^2}\,dx = \arctan(x)$.
   - Evaluate: $\frac{1}{\pi}[\arctan(x)]_0^1 = \frac{1}{\pi}(\arctan(1) - \arctan(0))$.
   - Substitute values: $\frac{1}{\pi}(\frac{\pi}{4} - 0) = \frac{1}{4}$.

   Answer: 0.25
