

Expectation II — linearity, the superpower

Lesson 30 · Module 2 — Discrete Random Variables · Probability & Statistics · dailymaths.uk/probstat

Practice problems

1. A hand of 5 cards is dealt from a standard 52-card deck. What is the expected number of aces in the hand?
2. You roll a fair 6-sided die 30 times. You win 2 dollars for every 6 rolled and lose 1 dollar for every 1 rolled. What are your expected total winnings?
3. In a group of 10 people, what is the expected number of pairs of people who share the same birth-month? (Assume each month is equally likely).

Solutions

1. A hand of 5 cards is dealt from a standard 52-card deck. What is the expected number of aces in the hand?

- Define I_i as the indicator variable that the i -th card is an ace for $i = 1, \dots, 5$.
- The probability that any single card is an ace is $P(I_i = 1) = \frac{4}{52} = \frac{1}{13}$.
- The expected value of each indicator is $E[I_i] = \frac{1}{13}$.
- By linearity of expectation, the total expected number of aces is $E[X] = \sum_{i=1}^5 E[I_i] = 5 \cdot \frac{1}{13} = \frac{5}{13}$.

Answer: $\frac{5}{13}$

2. You roll a fair 6-sided die 30 times. You win 2 dollars for every 6 rolled and lose 1 dollar for every 1 rolled. What are your expected total winnings?

- Let X be the number of 6s and Y be the number of 1s.
- The total winnings are $W = 2X - 1Y$.
- The expected number of 6s in 30 rolls is $E[X] = 30 \cdot \frac{1}{6} = 5$.
- The expected number of 1s in 30 rolls is $E[Y] = 30 \cdot \frac{1}{6} = 5$.
- By linearity, $E[W] = 2E[X] - 1E[Y] = 2(5) - 1(5) = 10 - 5 = 5$.

Answer: 5

3. In a group of 10 people, what is the expected number of pairs of people who share the same birth-month? (Assume each month is equally likely).

- There are $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ possible pairs of people.
- For any specific pair, the probability they share a birth-month is $\frac{12}{12 \cdot 12} = \frac{1}{12}$.
- Let I_{ij} be the indicator that pair (i, j) shares a month. $E[I_{ij}] = \frac{1}{12}$.
- The total number of pairs X is the sum of all 45 indicators.
- By linearity, $E[X] = 45 \cdot \frac{1}{12} = \frac{45}{12} = 3.75$.

Answer: 3.75