Bayes' theorem II — the medical test paradox

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Practice problems

1. A rare disease has a prevalence of 0.2%. A test for this disease has a sensitivity of 95% and a false positive rate of 3%. If a person tests positive, what is the probability they actually have the disease?
2. In a city, 1% of the residents are spies. A lie detector test is 90% accurate for spies (sensitivity) and 90% accurate for non-spies (specificity). If a resident tests as a spy, what is the probability they are actually a spy?
3. A security system triggers an alarm if it detects a breach. The probability of a breach is 0.01%. The system has a sensitivity of 99.9% and a false positive rate of 0.1%. If the alarm goes off, what is the probability there is actually a breach?

Solutions

1. A rare disease has a prevalence of 0.2%. A test for this disease has a sensitivity of 95% and a false positive rate of 3%. If a person tests positive, what is the probability they actually have the disease?

   - Identify priors: $P(D) = 0.002$ and $P(D^c) = 0.998$.
   - Identify likelihoods: $P(+|D) = 0.95$ and $P(+|D^c) = 0.03$.
   - Calculate total probability of positive test: $P(+) = (0.95 \times 0.002) + (0.03 \times 0.998) = 0.0019 + 0.02994 = 0.03184$.
   - Apply Bayes' theorem: $P(D|+) = \frac{0.0019}{0.03184} \approx 0.0597$.

   Answer: 0.0597

2. In a city, 1% of the residents are spies. A lie detector test is 90% accurate for spies (sensitivity) and 90% accurate for non-spies (specificity). If a resident tests as a spy, what is the probability they are actually a spy?

   - Priors: $P(S) = 0.01$, $P(S^c) = 0.99$.
   - Likelihoods: $P(+|S) = 0.90$. Since specificity is 90%, the false positive rate $P(+|S^c) = 1 - 0.90 = 0.10$.
   - Total probability of positive: $P(+) = (0.90 \times 0.01) + (0.10 \times 0.99) = 0.009 + 0.099 = 0.108$.
   - Posterior: $P(S|+) = \frac{0.009}{0.108} = \frac{1}{12} \approx 0.0833$.

   Answer: 0.0833

3. A security system triggers an alarm if it detects a breach. The probability of a breach is 0.01%. The system has a sensitivity of 99.9% and a false positive rate of 0.1%. If the alarm goes off, what is the probability there is actually a breach?

   - Priors: $P(B) = 0.0001$, $P(B^c) = 0.9999$.
   - Likelihoods: $P(+|B) = 0.999$, $P(+|B^c) = 0.001$.
   - Total probability of alarm: $P(+) = (0.999 \times 0.0001) + (0.001 \times 0.9999) = 0.0000999 + 0.0009999 = 0.0010998$.
   - Posterior: $P(B|+) = \frac{0.0000999}{0.0010998} \approx 0.0908$.

   Answer: 0.0908
