Inclusion-exclusion II — derangements

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Practice problems

1. Calculate the exact number of derangements for a set of 6 elements, $D(6)$.
2. A group of 5 people swap secret santa gifts. What is the probability that no one draws their own name?
3. In a set of 7 elements, how many permutations have exactly 2 elements in their original positions?

Solutions

1. Calculate the exact number of derangements for a set of 6 elements, $D(6)$.

   - Use the formula $D(6) = 6! \sum_{k=0}^{6} \frac{(-1)^k}{k!}$.
   - Expand the sum: $720 \times (1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720})$.
   - Simplify the terms: $720 \times (0 + 0.5 - 0.1666... + 0.04166... - 0.00833... + 0.00138...)$.
   - Calculate the sum inside: $720 \times (\frac{360-120+30-6+1}{720}) = 720 \times \frac{265}{720}$.
   - The result is 265.

   Answer: 265

2. A group of 5 people swap secret santa gifts. What is the probability that no one draws their own name?

   - The total number of permutations is $5! = 120$.
   - The number of derangements $D(5)$ is $120 \times (1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}) = 44$.
   - The probability is $\frac{D(5)}{5!} = \frac{44}{120}$.
   - Simplify the fraction: $\frac{11}{30}$.

   Answer: \frac1130

3. In a set of 7 elements, how many permutations have exactly 2 elements in their original positions?

   - First, choose which 2 elements are fixed: $\binom{7}{2} = \frac{7\times 6}{2} = 21$.
   - The remaining $7 - 2 = 5$ elements must be deranged so that no more elements are in their original positions.
   - The number of derangements for 5 elements is $D(5) = 44$.
   - The total number of such permutations is $\binom{7}{2} \times D(5) = 21 \times 44$.
   - Calculate $21 \times 44 = 924$.

   Answer: 924
