

Least squares I — the closest solution to an unsolvable system

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Practice problems

1. Find the least squares solution $\hat{x}$ for the system $Ax = b$ where $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.
2. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, find the least squares solution $\hat{x}$.
3. Explain why the normal equations $A^T A \hat{x} = A^T b$ are always consistent, even if $Ax = b$ is not.

Solutions

1. Find the least squares solution $\hat{x}$ for the system $Ax = b$ where $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

- Compute $A^T A = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = [1^2 + 2^2] = [5]$.
- Compute $A^T b = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = [1(1) + 2(4)] = [9]$.
- Solve the normal equation $5\hat{x} = 9$.
- Divide by 5 to get $\hat{x} = 1.8$.

Answer: $\hat{x} = 1.8$

2. Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, find the least squares solution $\hat{x}$.

- Compute $A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.
- Compute $A^T b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Solve $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Subtracting the equations or using the inverse gives $\hat{x}_1 = 1/3$ and $\hat{x}_2 = 1/3$.

Answer: $\hat{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$

3. Explain why the normal equations $A^T A \hat{x} = A^T b$ are always consistent, even if $Ax = b$ is not.
- The system $Ax = b$ is inconsistent if b is not in the column space of A .
 - The normal equations seek a vector $p = A\hat{x}$ such that p is the projection of b onto $Col(A)$.
 - The projection p always exists for any vector b and any subspace $Col(A)$.
 - Since p is by definition in $Col(A)$, there must exist some $\hat{x}$ such that $A\hat{x} = p$.
 - This $\hat{x}$ satisfies $A^T(b - A\hat{x}) = 0$, which is the normal equation.

Answer: Because the projection of b onto the column space of A always exists and is always reachable by a linear combination of the columns of A .