

The orthogonal complement; orthogonality of the four subspaces

Lesson 84 · Module 6 — Orthogonality & Least Squares · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Find a basis for the orthogonal complement $V^\perp$ of the subspace V in $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.
2. Given a 3×5 matrix A with rank 2, what is the dimension of the orthogonal complement of the row space of A ?
3. Let V be the subspace of $\mathbb{R}^4$ defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for $V^\perp$.

Solutions

1. Find a basis for the orthogonal complement $V^\perp$ of the subspace V in $\mathbb{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

- A vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in $V^\perp$ if its dot product with both spanning vectors is zero.
- This gives the system: $x + 2y + z = 0$ and $y - z = 0$.
- From the second equation, $y = z$.
- Substitute into the first: $x + 2z + z = 0$, so $x + 3z = 0$, which means $x = -3z$.
- The vectors are of the form $\begin{bmatrix} -3z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

2. Given a 3×5 matrix A with rank 2, what is the dimension of the orthogonal complement of the row space of A ?

- The row space of A is a subspace of $\mathbb{R}^5$ because the rows have 5 components.
- The dimension of the row space is equal to the rank of the matrix, which is 2.
- The dimension of the orthogonal complement $V^\perp$ is $n - \dim(V)$.
- Here $n = 5$ and $\dim(\text{Row}(A)) = 2$.
- Therefore, $\dim(\text{Row}(A)^\perp) = 5 - 2 = 3$.

Answer: 3

3. Let V be the subspace of $\mathbb{R}^4$ defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find a basis for $V^\perp$.

- The subspace V is defined as the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$.
- This means V is the null space of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.
- We know that $\text{Null}(A)^\perp = \text{Row}(A)$.
- The row space of A is spanned by the single row vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

- Thus, $V^\perp$ is spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$