

Orthogonal vectors and orthogonal subspaces

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Practice problems

1. Determine if the subspaces $V = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ and $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$ are orthogonal.
2. In $\mathbb{R}^4$, can two subspaces V and W both have dimension 3 and be orthogonal to each other?
3. Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find a basis for the row space and a basis for the null space, and verify they are orthogonal.

Solutions

1. Determine if the subspaces $V = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ and $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$ are orthogonal.

- Identify the basis for V as $v_1 = [1, 2, 1]^T$.
- Identify the basis for W as $w_1 = [1, 0, -1]^T$ and $w_2 = [0, 1, -2]^T$.
- Compute $v_1 \cdot w_1 = (1)(1) + (2)(0) + (1)(-1) = 1 + 0 - 1 = 0$.
- Compute $v_1 \cdot w_2 = (1)(0) + (2)(1) + (1)(-2) = 0 + 2 - 2 = 0$.
- Since all pairs of basis vectors are orthogonal, the subspaces are orthogonal.

Answer: Yes, they are orthogonal.

2. In $\mathbb{R}^4$, can two subspaces V and W both have dimension 3 and be orthogonal to each other?

- Recall the dimension constraint for orthogonal subspaces: $\dim(V) + \dim(W) \leq n$.
- Here, $n = 4$.
- The sum of dimensions is $3 + 3 = 6$.
- Since $6 > 4$, the condition is violated.

Answer: No, because the sum of their dimensions would exceed the dimension of the space.

3. Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, find a basis for the row space and a basis for the null space, and verify they are orthogonal.

- The row space is spanned by $[1, 2]^T$ since the second row is a multiple of the first.
- To find the null space, solve $x + 2y = 0$, which gives $x = -2y$. A basis vector is $[-2, 1]^T$.
- Compute the dot product: $(1)(-2) + (2)(1) = -2 + 2 = 0$.
- The basis vectors are orthogonal, so the subspaces are orthogonal.

Answer: Row space basis: $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, Null space basis: $\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$. Dot product is 0.