

Module 5 Review and Self-Test

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Practice problems

1. Find the standard matrix A for a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects a vector across the line $y = x$ and then scales the result by a factor of 2.
2. Given the matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, find the dimension of the kernel and the dimension of the image.
3. Determine if the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are similar by checking their invariants.

Solutions

1. Find the standard matrix A for a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects a vector across the line $y = x$ and then scales the result by a factor of 2.

- The reflection across $y = x$ maps $e_1 = [1, 0]^T$ to $[0, 1]^T$ and $e_2 = [0, 1]^T$ to $[1, 0]^T$.
- Applying the scaling factor of 2 to these results gives $T(e_1) = [0, 2]^T$ and $T(e_2) = [2, 0]^T$.
- The standard matrix A is formed by these images as columns.

Answer: $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

2. Given the matrix $A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, find the dimension of the kernel and the dimension of the image.

- The columns are identical, so the image is the span of $[1, 1]^T$, which has dimension 1.
- Using the Rank-Nullity Theorem, $\dim(\ker(A)) + \dim(\text{im}(A)) = 2$.
- Therefore, $\dim(\ker(A)) = 2 - 1 = 1$.

Answer: $\dim(\ker(A)) = 1$, $\dim(\text{im}(A)) = 1$

3. Determine if the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are similar by checking their invariants.

- Compute the determinant of A : $\det(A) = 1 \times 2 - 0 \times 0 = 2$.
- Compute the determinant of B : $\det(B) = 1 \times 2 - 1 \times 0 = 2$.
- Compute the trace of A : $\text{tr}(A) = 1 + 2 = 3$.
- Compute the trace of B : $\text{tr}(B) = 1 + 2 = 3$.
- Since both the determinant and trace match, the matrices are candidates for similarity, although matching invariants are necessary but not sufficient to prove similarity.

Answer: The invariants match, so they may be similar.