Transformations of function spaces: the derivative as a matrix

Practice problems

1. Given the polynomial $p(x) = 2 + 7x + 5x^2$, use the derivative matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ to find its derivative.
2. Compute the matrix D^2 for the space of polynomials of degree 2 or less, and explain what it represents.
3. Construct the derivative matrix D for the space of polynomials of degree 3 or less (basis: $1, x, x^2, x^3$).

Solutions

1. Given the polynomial $p(x) = 2 + 7x + 5x^2$, use the derivative matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ to find its derivative.

   - Step 1: Write the polynomial as a coordinate vector $\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}$.
   - Step 2: Multiply the matrix D by the vector $\mathbf{v}$.
   - Step 3: The first component is $0(2) + 1(7) + 0(5) = 7$.
   - Step 4: The second component is $0(2) + 0(7) + 2(5) = 10$.
   - Step 5: The third component is $0(2) + 0(7) + 0(5) = 0$.
   - Step 6: The resulting vector $\begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix}$ corresponds to the polynomial $7 + 10x$.

   Answer: 7 + 10x

2. Compute the matrix D^2 for the space of polynomials of degree 2 or less, and explain what it represents.

   - Step 1: Multiply D by itself: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
   - Step 2: Row 1 times Col 1 is 0, Row 1 times Col 2 is 0, Row 1 times Col 3 is $1(2) = 2$.
   - Step 3: Row 2 times all columns is 0, Row 3 times all columns is 0.
   - Step 4: The result is $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
   - Step 5: This matrix represents the second derivative operator.

   Answer: \beginbmatrix 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \endbmatrix

3. Construct the derivative matrix D for the space of polynomials of degree 3 or less (basis: $1, x, x^2, x^3$).

   - Step 1: Find the derivative of each basis element: $D(1) = 0, D(x) = 1, D(x^2) = 2x, D(x^3) = 3x^2$.
   - Step 2: Convert these to coordinate vectors: $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$.
   - Step 3: Place these vectors as columns in a 4×4 matrix.

   Answer: \beginbmatrix 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \endbmatrix