

The matrix of a transformation in a different basis — similarity

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Practice problems

1. Given the standard matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, find the matrix B' of the transformation relative to B .
2. If matrix A is similar to matrix B' , and $\det(A) = 5$, what is $\det(B')$? Explain using the properties of determinants.
3. Find a basis B such that the transformation $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is represented by the matrix $B' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Solutions

1. Given the standard matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, find the matrix B' of the transformation relative to B .

- Construct $P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- Compute $P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.
- Compute $AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
- Compute $B' = P^{-1}(AP) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Answer: $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

2. If matrix A is similar to matrix B' , and $\det(A) = 5$, what is $\det(B')$? Explain using the properties of determinants.
- Since A and B' are similar, $B' = P^{-1}AP$.
 - The determinant of a product is the product of the determinants: $\det(B') = \det(P^{-1})\det(A)\det(P)$.
 - Since $\det(P^{-1}) = 1/\det(P)$, these terms cancel out.
 - Therefore, $\det(B') = \det(A) = 5$.

Answer: 5

3. Find a basis B such that the transformation $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is represented by the matrix $B' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

- We need P such that $AP = PB'$.
- This means $Av_1 = 1v_1$ and $Av_2 = -1v_2$, where v_1 and v_2 are the columns of P .
- Solving $Av_1 = v_1$ for $v_1 = [x, y]^T$ gives $y = x$, so $v_1 = [1, 1]^T$ is a solution.
- Solving $Av_2 = -v_2$ for $v_2 = [x, y]^T$ gives $y = -x$, so $v_2 = [1, -1]^T$ is a solution.
- One such basis is $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

Answer: $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$