

Linear maps in 3D; projections onto planes

Lesson 77 · Module 5 — Linear Transformations · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Find the standard matrix A for the orthogonal projection onto the xz -plane.
2. Given a plane with normal vector $\mathbf{n} = [0, 1, 1]^T$, find the projection of the vector $\mathbf{x} = [2, 3, 4]^T$ onto the plane.
3. Prove that for any projection matrix P onto a plane, $P^2 = P$.

Solutions

1. Find the standard matrix A for the orthogonal projection onto the xz -plane.

- The normal vector to the xz -plane is $\mathbf{n} = [0, 1, 0]^T$.
- The projection of $\mathbf{e}_1 = [1, 0, 0]^T$ onto the plane is $[1, 0, 0]^T$ because it already lies in the plane.
- The projection of $\mathbf{e}_2 = [0, 1, 0]^T$ onto the plane is the zero vector because it is the normal vector.
- The projection of $\mathbf{e}_3 = [0, 0, 1]^T$ onto the plane is $[0, 0, 1]^T$ because it already lies in the plane.
- Assembling these as columns gives the matrix.

Answer: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Given a plane with normal vector $\mathbf{n} = [0, 1, 1]^T$, find the projection of the vector $\mathbf{x} = [2, 3, 4]^T$ onto the plane.

- Compute $\mathbf{x} \cdot \mathbf{n} = (2)(0) + (3)(1) + (4)(1) = 7$.
- Compute $\mathbf{n} \cdot \mathbf{n} = 0^2 + 1^2 + 1^2 = 2$.
- The projection onto the normal is $\text{Proj}_{\mathbf{n}}(\mathbf{x}) = \frac{7}{2}[0, 1, 1]^T = [0, 3.5, 3.5]^T$.
- The projection onto the plane is $\mathbf{x} - \text{Proj}_{\mathbf{n}}(\mathbf{x}) = [2, 3, 4]^T - [0, 3.5, 3.5]^T = [2, -0.5, 0.5]^T$.

Answer: $[2, -0.5, 0.5]^T$

3. Prove that for any projection matrix P onto a plane, $P^2 = P$.

- Geometrically, $P\mathbf{x}$ is the projection of $\mathbf{x}$ onto the plane W .
- The resulting vector $P\mathbf{x}$ already lies within the plane W .
- Projecting a vector that is already in the plane does not change it, so $P(P\mathbf{x}) = P\mathbf{x}$.
- Since this holds for all $\mathbf{x}$, we have $P^2 = P$.

Answer: $P^2 = P$