

Invertible transformations — undoing a map

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Practice problems

1. Determine if the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ is invertible. If it is, find A^{-1} .
2. A linear transformation T is represented by $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. Explain why T is not invertible and describe the geometry of its image.
3. Given the transformation $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find the vector $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solutions

1. Determine if the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ is invertible. If it is, find A^{-1} .
- Calculate the determinant: $\det(A) = (1)(7) - (3)(2) = 7 - 6 = 1$.
 - Since $\det(A) \neq 0$, the matrix is invertible.
 - Apply the 2x2 inverse formula: swap diagonal elements (7 and 1), negate off-diagonal elements (-3 and -2), and divide by the determinant (1).
 - The result is $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

Answer: $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

2. A linear transformation T is represented by $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$. Explain why T is not invertible and describe the geometry of its image.
- Calculate the determinant: $\det(A) = (2)(2) - (4)(1) = 4 - 4 = 0$.
 - Because the determinant is zero, the transformation collapses the plane, making it non-invertible.
 - Observe that the second column $[4, 2]^T$ is exactly twice the first column $[2, 1]^T$.
 - Therefore, the image of T is the line spanned by the vector $[2, 1]^T$.

Answer: Not invertible because $\det(A) = 0$; the image is a line.

3. Given the transformation $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find the vector $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
- Identify A as a 90-degree counter-clockwise rotation. Its inverse A^{-1} should be a 90-degree clockwise rotation.
 - Compute $\det(A) = (0)(0) - (-1)(1) = 1$.
 - Find $A^{-1} = \frac{1}{1} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
 - Multiply A^{-1} by the output vector: $\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} (0)(3) + (1)(-2) \\ (-1)(3) + (0)(-2) \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

Answer: $\mathbf{x} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$