

Kernel and image; injectivity and surjectivity

Lesson 75 · Module 5 — Linear Transformations · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Find the kernel and image of the transformation represented by $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Is the map injective? Is it surjective?
2. A linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by $T([x, y]^T) = [x + y, x - y, 2x]^T$. Determine if T is injective.
3. If $T : V \rightarrow W$ is a linear transformation where $\dim(V) = 4$ and $\dim(\ker(T)) = 1$, is T surjective if $\dim(W) = 3$?

Solutions

1. Find the kernel and image of the transformation represented by $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Is the map injective? Is it surjective?

- Solve $A\mathbf{x} = \mathbf{0}$ using row reduction. $R_3 \leftarrow R_3 - R_1$ gives $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, then $R_3 \leftarrow R_3 - R_2$

gives $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

- From R_2 , $x_2 = -x_3$. From R_1 , $x_1 = -2x_2 - x_3 = -2(-x_3) - x_3 = x_3$.
- The kernel is $\text{span}\{[1, -1, 1]^T\}$. Since it is non-trivial, the map is not injective.
- The rank is 2 (two pivots). Since the target space is $\mathbb{R}^3$ and $2 < 3$, the map is not surjective.

Answer: $\ker(T) = \text{span} \{ [1, -1, 1]^T \}$, $\text{im}(T) = \text{span} \{ [1, 0, 1]^T, [2, 1, 3]^T \}$, not injective, not surjective

2. A linear map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is defined by $T([x, y]^T) = [x + y, x - y, 2x]^T$. Determine if T is injective.

- Set $T([x, y]^T) = [0, 0, 0]^T$.
- This gives the system: $x + y = 0$, $x - y = 0$, and $2x = 0$.
- From $2x = 0$, we have $x = 0$. Substituting into $x + y = 0$ gives $y = 0$.
- The only solution is the zero vector $[0, 0]^T$.
- Since the kernel is trivial, the map is injective.

Answer: Yes, it is injective.

3. If $T : V \rightarrow W$ is a linear transformation where $\dim(V) = 4$ and $\dim(\ker(T)) = 1$, is T surjective if $\dim(W) = 3$?

- Use the Rank-Nullity Theorem: $\dim(V) = \text{rank}(T) + \dim(\ker(T))$.
- Substitute the known values: $4 = \text{rank}(T) + 1$.
- This implies $\text{rank}(T) = 3$.
- The dimension of the image is 3, and the dimension of the target space W is also 3.
- Since $\text{im}(T) = W$, the map is surjective.

Answer: Yes, it is surjective.