

Linear transformations: functions that respect the grid

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Practice problems

1. Given the linear transformation matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, find the image of the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.
2. A linear transformation T maps $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the image of $\mathbf{w} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
3. Determine if the function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ is a linear transformation. Justify your answer using the scaling rule.

Solutions

1. Given the linear transformation matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, find the image of the vector $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

- Multiply the matrix A by the vector $\mathbf{v}$ using the column picture.
- The result is $5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$.
- Compute the components: $5(2) + 2(-3) = 10 - 6 = 4$ and $5(1) + 2(4) = 5 + 8 = 13$.

Answer: $\begin{bmatrix} 4 \\ 13 \end{bmatrix}$

2. A linear transformation T maps $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the image of $\mathbf{w} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

- Construct the matrix A using the images of the basis vectors as columns: $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$.
- Multiply A by $\mathbf{w}$: $\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
- Compute the components: $0(3) + (-1)(-2) = 2$ and $2(3) + 3(-2) = 6 - 6 = 0$.

Answer: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

3. Determine if the function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ is a linear transformation. Justify your answer using the scaling rule.

- Test the homogeneity rule $T(c\mathbf{v}) = cT(\mathbf{v})$.
- Let $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $c = 2$.
- Compute $T(2\mathbf{v}) = T \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.
- Compute $2T(\mathbf{v}) = 2T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.
- Since $\begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, the rule is violated.

Answer: No, it is not a linear transformation.