

Module 4 Review and Self-Test

Lesson 70 · Module 4 — Vector Spaces · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Let S be the subspace of $\mathbb{R}^3$ defined by $x+y+z = 0$. Determine if the set $B = \{[-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a basis for S .
2. Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.
3. Given basis $B_1 = \{[1, 0]^T, [0, 1]^T\}$ and $B_2 = \{[1, 1]^T, [1, -1]^T\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.

Solutions

1. Let S be the subspace of $\mathbb{R}^3$ defined by $x+y+z=0$. Determine if the set $B = \{[-1, 1, 0]^T, [-1, 0, 1]^T\}$ is a basis for S .
- First, check if the vectors are in S . For $[-1, 1, 0]^T$, $-1 + 1 + 0 = 0$. For $[-1, 0, 1]^T$, $-1 + 0 + 1 = 0$. Both are in S .
 - Check for linear independence. Since neither vector is a scalar multiple of the other, they are linearly independent.
 - Determine the dimension of S . S is a plane in $\mathbb{R}^3$ defined by one equation, so $\dim(S) = 3 - 1 = 2$.
 - Since we have 2 linearly independent vectors in a 2-dimensional space, they must form a basis.

Answer: Yes, it is a basis.

2. Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$.

- Perform row reduction. $R_3 \leftarrow R_3 - R_1$ gives $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
- Then $R_3 \leftarrow R_3 - R_2$ gives $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
- There are 2 pivots, so $\text{rank}(A) = 2$.
- The matrix has 3 columns, so $\text{nullity}(A) = 3 - 2 = 1$.

Answer: $\text{rank}=2, \text{nullity}=1$

3. Given basis $B_1 = \{[1, 0]^T, [0, 1]^T\}$ and $B_2 = \{[1, 1]^T, [1, -1]^T\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.
- We need to express vectors of B_1 in terms of B_2 .
 - For $[1, 0]^T$: $c_1[1, 1]^T + c_2[1, -1]^T = [1, 0]^T$. This gives $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$, so $c_1 = 0.5, c_2 = 0.5$.
 - For $[0, 1]^T$: $c_1[1, 1]^T + c_2[1, -1]^T = [0, 1]^T$. This gives $c_1 + c_2 = 0$ and $c_1 - c_2 = 1$, so $c_1 = 0.5, c_2 = -0.5$.
 - The matrix P is formed by these coordinate vectors as columns.

Answer: $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$