

Function spaces and other exotic vector spaces

Lesson 69 · Module 4 — Vector Spaces · Linear Algebra · dailymaths.uk/linalg

Practice problems

1. Determine if the polynomials $p_1 = 1 + x^2$, $p_2 = x + x^2$, and $p_3 = 1 + x$ are linearly independent in the space of polynomials of degree 2 or less.
2. Find the coordinates of the polynomial $p(x) = 4 + 2x - 3x^2$ relative to the basis $B = \{1, 1 + x, 1 + x + x^2\}$.
3. Consider the space of polynomials of degree 3 or less. Is the set $\{1 + x, x + x^2, x^2 + x^3, 1 + x^3\}$ linearly independent?

Solutions

1. Determine if the polynomials $p_1 = 1 + x^2$, $p_2 = x + x^2$, and $p_3 = 1 + x$ are linearly independent in the space of polynomials of degree 2 or less.

- Convert polynomials to coordinate vectors relative to basis $\{1, x, x^2\}$: $[p_1] = [1, 0, 1]^T$, $[p_2] = [0, 1, 1]^T$, $[p_3] = [1, 1, 0]^T$.
- Form matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.
- Perform row reduction: $R_3 \leftarrow R_3 - R_1$ gives $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.
- Then $R_3 \leftarrow R_3 - R_2$ gives $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.
- Since there are 3 pivots, the vectors are linearly independent.

Answer: Yes, they are linearly independent.

2. Find the coordinates of the polynomial $p(x) = 4 + 2x - 3x^2$ relative to the basis $B = \{1, 1 + x, 1 + x + x^2\}$.

- Set up the equation $4 + 2x - 3x^2 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)$.
- Equate coefficients of x^2 : $c_3 = -3$.
- Equate coefficients of x : $c_2 + c_3 = 2 \implies c_2 - 3 = 2 \implies c_2 = 5$.
- Equate constants: $c_1 + c_2 + c_3 = 4 \implies c_1 + 5 - 3 = 4 \implies c_1 + 2 = 4 \implies c_1 = 2$.
- The coordinate vector is $[2, 5, -3]^T$.

Answer: $[2, 5, -3]^T$

3. Consider the space of polynomials of degree 3 or less. Is the set $\{1 + x, x + x^2, x^2 + x^3, 1 + x^3\}$ linearly independent?

- Convert to coordinate vectors in $\mathbb{R}^4$: $v_1 = [1, 1, 0, 0]^T$, $v_2 = [0, 1, 1, 0]^T$, $v_3 = [0, 0, 1, 1]^T$, $v_4 = [1, 0, 0, 1]^T$.
- Form matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.
- Row reduce: $R_2 \leftarrow R_2 - R_1$ gives $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.
- Then $R_3 \leftarrow R_3 - R_2$ gives $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

- Then $R_4 \leftarrow R_4 - R_3$ gives $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

- There are only 3 pivots for 4 vectors, so they are linearly dependent.

Answer: No, they are linearly dependent.