

Change of basis II — worked examples

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Practice problems

1. Given basis $B_1 = \{[2, 1]^T, [1, 2]^T\}$ and the standard basis $B_2 = \{[1, 0]^T, [0, 1]^T\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.
2. Let $B_1 = \{[1, 1]^T, [0, 1]^T\}$ and $B_2 = \{[1, 0]^T, [1, 1]^T\}$. Find the change of basis matrix $P_{B_2 \leftarrow B_1}$.
3. A vector v has coordinates $[v]_{B_1} = [3, -2]^T$ relative to $B_1 = \{[1, 2]^T, [3, 4]^T\}$. Find its coordinates relative to $B_2 = \{[1, 0]^T, [0, 1]^T\}$.

Solutions

1. Given basis $B_1 = \{[2, 1]^T, [1, 2]^T\}$ and the standard basis $B_2 = \{[1, 0]^T, [0, 1]^T\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.

- The matrix P_1 has columns $[2, 1]^T$ and $[1, 2]^T$.
- The matrix P_2 is the identity matrix I .
- The formula is $P_{B_2 \leftarrow B_1} = P_2^{-1}P_1 = I^{-1}P_1 = P_1$.
- Therefore, the matrix is simply the matrix of B_1 vectors.

Answer: $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

2. Let $B_1 = \{[1, 1]^T, [0, 1]^T\}$ and $B_2 = \{[1, 0]^T, [1, 1]^T\}$. Find the change of basis matrix $P_{B_2 \leftarrow B_1}$.

- Construct $P_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
- Compute P_2^{-1} . The determinant is $1(1) - 1(0) = 1$. The inverse is $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.
- Multiply $P_2^{-1}P_1$: $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 0-1 \\ 0+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

Answer: $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

3. A vector v has coordinates $[v]_{B_1} = [3, -2]^T$ relative to $B_1 = \{[1, 2]^T, [3, 4]^T\}$. Find its coordinates relative to $B_2 = \{[1, 0]^T, [0, 1]^T\}$.

- The change of basis matrix from B_1 to the standard basis B_2 is $P_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.
- The coordinates in B_2 are found by $[v]_{B_2} = P_1[v]_{B_1}$.
- Compute the product: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(3) + 3(-2) \\ 2(3) + 4(-2) \end{bmatrix} = \begin{bmatrix} 3-6 \\ 6-8 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$.

Answer: $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$