Change of basis I — the idea and the matrix

Practice problems

1. Given basis $B_1 = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}\}$ and basis $B_2 = \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.

2. If the change of basis matrix from B_1 to B_2 is $P = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$, and a vector v has coordinates $[v]_{B_1} = \begin{bmatrix}1\\-1\end{bmatrix}$, find $[v]_{B_2}$.

3. Let $P_1 = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $P_2 = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$ be matrices whose columns are the bases B_1 and B_2 respectively. Find the change of basis matrix $P_{B_1 \leftarrow B_2}$.

Solutions

1. Given basis $B_1 = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}\}$ and basis $B_2 = \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$, find the change of basis matrix $P_{B_2 \leftarrow B_1}$.

   - Find coordinates of $\begin{bmatrix}1\\2\end{bmatrix}$ relative to B_2: $1\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$, so the first column is $\begin{bmatrix}1\\1\end{bmatrix}$.
   - Find coordinates of $\begin{bmatrix}3\\5\end{bmatrix}$ relative to B_2: $3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$, so the second column is $\begin{bmatrix}3\\2\end{bmatrix}$.
   - Assemble the matrix.

   Answer: \beginbmatrix 1 & 3 \\ 1 & 2 \endbmatrix

2. If the change of basis matrix from B_1 to B_2 is $P = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$, and a vector v has coordinates $[v]_{B_1} = \begin{bmatrix}1\\-1\end{bmatrix}$, find $[v]_{B_2}$.

   - Use the formula $[v]_{B_2} = P_{B_2 \leftarrow B_1}[v]_{B_1}$.
   - Compute the product $\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$.
   - The first component is $2(1) + 1(-1) = 1$.
   - The second component is $0(1) + 3(-1) = -3$.

   Answer: \beginbmatrix 1 \\ -3 \endbmatrix

3. Let $P_1 = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $P_2 = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$ be matrices whose columns are the bases B_1 and B_2 respectively. Find the change of basis matrix $P_{B_1 \leftarrow B_2}$.

   - The matrix from B_2 to B_1 is $P_{B_1 \leftarrow B_2} = P_1^{-1}P_2$.
   - Compute P_1^{-1}: for $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$, the inverse is $\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$.
   - Multiply P_1^{-1} by P_2: $\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$.
   - Top left: $1(2) + (-1)(1) = 1$. Top right: $1(0) + (-1)(1) = -1$.
   - Bottom left: $0(2) + 1(1) = 1$. Bottom right: $0(0) + 1(1) = 1$.

   Answer: \beginbmatrix 1 & -1 \\ 1 & 1 \endbmatrix