

The four fundamental subspaces (the big picture)

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Practice problems

1. Given the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{bmatrix}$, find the dimensions of $C(A)$, $C(A^T)$, $N(A)$, and $N(A^T)$.
2. A matrix A has 7 rows and 12 columns. If the dimension of the null space is 5, what are the dimensions of the other three fundamental subspaces?
3. If A is a square $n \times n$ matrix and the left null space $N(A^T)$ contains only the zero vector, what can you conclude about the other three subspaces?

Solutions

1. Given the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 \end{bmatrix}$, find the dimensions of $C(A)$, $C(A^T)$, $N(A)$, and $N(A^T)$.

- Perform row reduction: $R3 \leftarrow R3 - R1$ gives $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.
- Further reduction: $R3 \leftarrow R3 - R2$ gives $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
- There are 2 pivots, so $\text{rank}(A) = r = 2$.
- The matrix has $m = 3$ rows and $n = 4$ columns.
- $\dim(C(A)) = r = 2$ and $\dim(C(A^T)) = r = 2$.
- $\dim(N(A)) = n - r = 4 - 2 = 2$ and $\dim(N(A^T)) = m - r = 3 - 2 = 1$.

Answer: $\dim(C(A))=2$, $\dim(C(A^T))=2$, $\dim(N(A))=2$, $\dim(N(A^T))=1$

2. A matrix A has 7 rows and 12 columns. If the dimension of the null space is 5, what are the dimensions of the other three fundamental subspaces?

- We are given $n = 12$, $m = 7$, and $\dim(N(A)) = 5$.
- Using the rank-nullity theorem: $r + 5 = 12$, so $r = 7$.
- The dimension of the column space $C(A)$ is $r = 7$.
- The dimension of the row space $C(A^T)$ is $r = 7$.
- The dimension of the left null space $N(A^T)$ is $m - r = 7 - 7 = 0$.

Answer: $\dim(C(A))=7$, $\dim(C(A^T))=7$, $\dim(N(A^T))=0$

3. If A is a square $n \times n$ matrix and the left null space $N(A^T)$ contains only the zero vector, what can you conclude about the other three subspaces?

- If $N(A^T) = \{0\}$, then $\dim(N(A^T)) = 0$.
- Since $\dim(N(A^T)) = m - r$ and $m = n$, we have $n - r = 0$, which means $r = n$.
- Since $r = n$, the column space $C(A)$ has dimension n , meaning $C(A) = \mathbb{R}^n$.
- The row space $C(A^T)$ also has dimension n , meaning $C(A^T) = \mathbb{R}^n$.
- The null space $N(A)$ has dimension $n - r = n - n = 0$, meaning $N(A) = \{0\}$.

Answer: All four subspaces are determined: $C(A) = \mathbb{R}^n$, $C(A^T) = \mathbb{R}^n$, $N(A) = \{0\}$, $N(A^T) = \{0\}$