

# Finding bases for null space and column space

Lesson 62 · Module 4 — Vector Spaces · Linear Algebra · [dailymaths.uk/linalg](http://dailymaths.uk/linalg)

---

## Practice problems

1. Find a basis for the column space and the null space of the matrix  $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 6 & 0 \\ 1 & 3 & -1 \end{pmatrix}$ .
2. Given a  $4 \times 4$  matrix  $A$  with  $\text{rank}(A) = 2$ , how many vectors will be in a basis for the null space? Explain your reasoning.
3. Find a basis for the null space of  $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ .

## Solutions

- Find a basis for the column space and the null space of the matrix  $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 6 & 0 \\ 1 & 3 & -1 \end{pmatrix}$ .
  - Perform row reduction:  $R2 \rightarrow R2 - 2R1$  and  $R3 \rightarrow R3 - R1$  gives  $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix}$ .
  - Further reduce to RREF:  $R2 \rightarrow -0.5R2$  and  $R3 \rightarrow R3 - R2$  and  $R1 \rightarrow R1 - R2$  gives  $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ .
  - Pivots are in columns 1 and 3. The basis for  $\text{Col}(A)$  is the 1st and 3rd columns of the original matrix:  $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ .
  - For the null space,  $x_2$  is free.  $x_1 + 3x_2 = 0 \implies x_1 = -3x_2$  and  $x_3 = 0$ .
  - Setting  $x_2 = 1$  gives the special solution  $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ . This is the basis for  $\text{Null}(A)$ .

**Answer:** \text{Col basis: } \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}, \text{Null basis: } \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}

- Given a  $4 \times 4$  matrix  $A$  with  $\text{rank}(A) = 2$ , how many vectors will be in a basis for the null space? Explain your reasoning.
  - The rank of a matrix is the dimension of its column space, so  $\dim(\text{Col } A) = 2$ .
  - The rank-nullity theorem states that  $\dim(\text{Col } A) + \dim(\text{Null } A) = n$ , where  $n$  is the number of columns.
  - Here  $n = 4$ , so  $2 + \dim(\text{Null } A) = 4$ .
  - Therefore,  $\dim(\text{Null } A) = 2$ .

**Answer:** 2 vectors

- Find a basis for the null space of  $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ .
  - Reduce to RREF:  $R2 \rightarrow R2 - 2R1$  gives  $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ .
  - The pivot is in column 1, so  $x_2$  is a free variable.
  - The equation is  $x_1 + 2x_2 = 0$ , which means  $x_1 = -2x_2$ .
  - Setting  $x_2 = 1$  gives the vector  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ .

**Answer:**  $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$